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The lattice of subracks is atomic

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ABSTRACT

A rack is a set together with a self-distributive bijective binary operation. In this paper, we give a positive answer to a question due to Heckenberger, Shareshian and Welker. Indeed, we prove that the lattice of subracks of a rack is atomic. Further, by using the atoms, we associate certain quandles to racks. We also show that the lattice of subracks of a rack is isomorphic to the lattice of subracks of a quandle. Moreover, we show that the lattice of subracks of a rack is distributive if and only if its corresponding quandle is the trivial quandle. So the lattice of subracks of a rack is distributive if and only if it is a Boolean lattice.

1. Introduction

In 1943, a certain algebraic structure, known as *key* or *involutory quandle*, was introduced by M. Takasaki in [7] to study the notion of reflection in the context of finite geometry. In 1959, J.C. Conway and G.C. Wraith introduced a more general algebraic structure called *wrack* in an unpublished correspondence. In 1982, D. Joyce for the first

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time used the word *quandle* for an algebraic and combinatorial structure to study *knot invariants* [5]. Joyce's definition of quandle is the same as the one which is nowadays used.

Let R be a set together with a binary operation $\triangleright$ which satisfies the equality $a \triangleright (b \triangleright c) = (a \triangleright b) \triangleright (a \triangleright c)$, for all $a, b, c \in R$. This equality is called (*left*) *self-distributivity* identity. A *knot* is an embedding of S^1 in $\mathbb{R}^3$. In 1984, S. Matveev, and in 1986, E. Brieskorn independently used self-distributivity systems to study the isotopy type of braids and knots, in [6] and [2], respectively. In 1992, R. Fenn and C. Rourke initiated to use the word *rack* instead of wrack. They used racks to study links and knots in 3-manifolds [3]. A rack is indeed a generalization of the concept of quandle. Racks are used to encode the movements of knots and links in the space.

In the following, the definition of a rack and some known examples of racks are given.

Definition 1.1. A *rack* R is a set together with a binary operation $\triangleright$ such that

(1) for all a, b and c in R, $a \triangleright (b \triangleright c) = (a \triangleright b) \triangleright (a \triangleright c)$, and
(2) for all a and b in R there exists a unique $c \in R$ with $a \triangleright c = b$.

Conditions (1) and (2) are called self-distributivity and *bijectivity*, respectively. A rack R is called a *quandle* if it satisfies the following additional condition:

$$a \triangleright a = a, \quad \text{for all } a \in R.$$

It follows from the bijectivity condition of racks that the function $f_a : R \to R$ with $f_a(b) = a \triangleright b$ is bijective, for all $a \in R$. Therefore, by self-distributivity we have $f_a(b) \triangleright f_a(c) = f_a f_b(c)$, for all $a, b, c \in R$.

Example 1.2. The followings are some known examples of racks:

(1) Let R be a set and $a \triangleright b = b$, for all $a, b \in R$. Then R is a quandle, called the *trivial quandle*.
(2) Let R be a set and f be a permutation on R. Define $a \triangleright b = f(b)$, for all $a, b \in R$. Then R is a rack, but not a quandle.
(3) Let A be an abelian group and $a \triangleright b = 2a - b$, for all $a, b \in A$. Then A is a quandle, called the *dihedral quandle*.
(4) Let G be a group and $a \triangleright b = ab^{-1}a$, for all $a, b \in G$. Then G is a quandle, called the *core quandle* (or *rack*).
(5) Let $S = \mathbb{Z}[t, t^{-1}]$ be the ring of Laurent polynomials with integer coefficients, and M be an S-module. Define $a \triangleright b = (1 - t)a + tb$, for all $a, b \in M$. Then M is a quandle, called the *Alexander quandle*.
(6) Let $S = \mathbb{Z}[t, t^{-1}, s]$ be the ring of all polynomials over $\mathbb{Z}$ with the variables s, t, t^{-1} such that t is invertible with the inverse t^{-1}. Assume that $R = S/\left\langle s^2 - s(1 - t)\right\rangle$,