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# Opposition diagrams for automorphisms of large spherical buildings

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ABSTRACT

Let $\theta$ be an automorphism of a thick irreducible spherical building $\Delta$ of rank at least 3 with no Fano plane residues. We prove that if there exist both type $J_1$ and $J_2$ simplices of $\Delta$ mapped onto opposite simplices by $\theta$, then there exists a type $J_1 \cup J_2$ simplex of $\Delta$ mapped onto an opposite simplex by $\theta$. This property is called *cappedness*. We give applications of cappedness to opposition diagrams, domesticity, and the calculation of displacement in spherical buildings. In a companion piece to this paper we study the thick irreducible spherical buildings containing Fano plane residues. In these buildings automorphisms are not necessarily capped.



## 0. Introduction

Let $\theta$ be an automorphism of a spherical building $\Delta$ of type $(W, S)$. The analysis of the fixed element geometry $\mathrm{Fix}(\theta)$ of $\theta$ is a powerful and well-established technique in building theory, see for example the beautiful theory of Tits indices and fixed subbuildings [12,18]. A complementary concept to fixed element theory is the "opposite geometry" $\mathrm{Opp}(\theta)$ consisting of all simplices of $\Delta$ that are mapped onto opposite simplices by $\theta$. This geometry arises naturally in Curtis–Phan Theory, where it is used to efficiently

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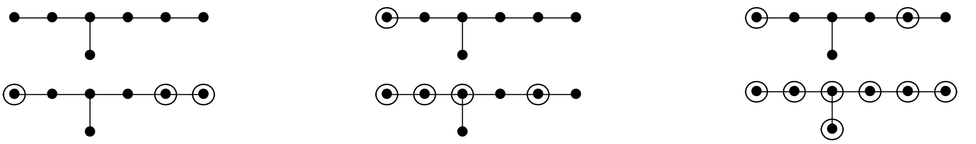
encode presentations of groups acting on buildings (see [3,7]). However compared to the fixed element theory very little is known concerning $\mathrm{Opp}(\theta)$. In this paper we initiate a systematic analysis of the structure of the geometry $\mathrm{Opp}(\theta)$.

To motivate and illustrate the key concepts in an example, let $\theta$ be a collineation of a thick $\mathsf{E}_7$ building $\Delta$, and construct the *opposition diagram* of $\theta$ by encircling all nodes $s \in S$ of the Coxeter graph with the property that there exists a type $s$ vertex in $\mathrm{Opp}(\theta)$. What are the possible opposition diagrams that can arise? It turns out that the number of possible diagrams is far less than the trivial bound of $2^7$. In fact it follows from our work that there are only 6 possibilities:



A fundamental result of Leeb [8, §5] and Abramenko and Brown [2, Proposition 4.2] states that if $\theta$ is a nontrivial automorphism of a thick spherical building then $\mathrm{Opp}(\theta)$ is necessarily nonempty, and hence the first diagram above occurs if and only if $\theta$ is the identity. For the second, third, forth, and fifth diagrams it is clear that the automorphism in question maps no chamber to an opposite chamber. Automorphisms mapping no chamber to an opposite chamber are called *domestic automorphisms* (the terminology here is aligned with the thematics of the language of building theory, reflecting the idea that these automorphisms stay "close to home"). These automorphisms have recently enjoyed extensive investigation, including the series [15–17] where domesticity in projective spaces, polar spaces, and generalised quadrangles is studied, [20] where symplectic polarities of large $\mathsf{E}_6$ buildings are classified in terms of domesticity, [21] where domestic trialities of $\mathsf{D}_4$ buildings are classified, and [9] where domesticity in generalised polygons is studied.

Returning to the $\mathsf{E}_7$ example, if $\theta$ is not domestic then the opposition diagram of $\theta$ is necessarily the sixth of the above diagrams, with all nodes encircled. However, can this diagram be the opposition diagram of a domestic automorphism? It is a priori possible that there are vertices of each type $1, 2, 3, 4, 5, 6$ and $7$ mapped onto opposite vertices, yet no chamber mapped to an opposite chamber. Such an automorphism is called *exceptional domestic*. It turns out from the results of this paper that if the $\mathsf{E}_7$ building $\Delta$ contains no Fano plane residues then exceptional domestic automorphisms do not exist. In contrast, we show in [10] that if $\Delta$ is an $\mathsf{E}_7$ building containing a Fano residue (thus $\Delta$ is the building of the Chevalley group $\mathsf{E}_7(2)$) then $\Delta$ admits exceptional domestic automorphisms.

More generally one may ask whether the existence of both a type $J_1$ simplex and a type $J_2$ simplex in $\mathrm{Opp}(\theta)$ implies the existence of a type $J_1 \cup J_2$ simplex in $\mathrm{Opp}(\theta)$. An automorphism satisfying this property is called *capped*. An equivalent formulation of this concept is as follows. The *type* $\mathrm{Typ}(\theta)$ of an automorphism $\theta$ is the union of all subsets