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Subprime solutions of the classical Yang–Baxter equation

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ABSTRACT

We introduce a new family of classical r-matrices for the Lie algebra $\mathfrak{sl}_n$ that lies in the Zariski boundary of the Belavin–Drinfeld space $\mathcal{M}$ of quasi-triangular solutions to the classical Yang–Baxter equation. In this setting $\mathcal{M}$ is a finite disjoint union of components; exactly $\phi(n)$ of these components are SL_n-orbits of single points. These points are the generalized Cremmer–Gervais r-matrices $r_{i,n}$ which are naturally indexed by pairs of positive coprime integers, i and n, with $i < n$. A conjecture of Gerstenhaber and Giaquinto states that the boundaries of the Cremmer–Gervais components contain r-matrices having maximal parabolic subalgebras $\mathfrak{p}_{i,n} \subseteq \mathfrak{sl}_n$ as carriers. We prove this conjecture in the cases when $n \equiv \pm 1 \pmod{i}$. The subprime linear functionals $f \in \mathfrak{p}_{i,n}^*$ and the corresponding principal elements $H \in \mathfrak{p}_{i,n}$ play important roles in our proof. Since the subprime functionals are Frobenius precisely in the cases when $n \equiv \pm 1 \pmod{i}$, this partly explains our need to require these conditions on i and n. We conclude with a proof of the GG boundary conjecture in an unrelated case, namely when $(i, n) = (5, 12)$, where the subprime functional is no longer a Frobenius functional.

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1. Introduction and main results

Throughout this paper, we assume the ground field $\mathbb{F}$ has characteristic 0 although most results in this paper also hold for fields of nearly any other characteristic. In particular, we will fix a pair of positive integers i and n with $i < n$, and in some of the calculations that follow, the numbers 2 and n appear in denominators. Thus we need $2n$ to be a nonzero element of the ground field $\mathbb{F}$. For vectors u, v in an $\mathbb{F}$-vector space V, define $u \wedge v := \frac{1}{2}(u \otimes v - v \otimes u) \in V \wedge V \subseteq V \otimes V$. For a Lie algebra $\mathfrak{g}$ and an element in the tensor space $r = \sum a_i \wedge b_i \in \mathfrak{g} \wedge \mathfrak{g}$ we say that r is a classical r-matrix if the Schouten bracket of r with itself

$$\langle r, r\rangle := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \tag{1.1}$$

is $\mathfrak{g}$-invariant. Here, $r_{12} = r \otimes 1$, $r_{23} = 1 \otimes r$, and $r_{13} = \sigma(r_{23})$, where σ is the linear endomorphism of $\mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ that permutes the first two tensor components: $\sigma(x \otimes y \otimes z) = y \otimes x \otimes z$. Classical r-matrices arise naturally in the context of Poisson–Lie groups and Lie bialgebras (see e.g. [2, Chapter 1]). If $\langle r, r\rangle = 0$, then r is said to be a solution to the classical Yang–Baxter equation (CYBE). On the other hand, if $\langle r, r\rangle$ is non-zero and $\mathfrak{g}$-invariant, r is said to be a solution to the modified classical Yang–Baxter equation (MCYBE). Following [8], we let $\mathcal{C}$ and $\mathcal{M}$ denote the solution spaces of the CYBE and MCYBE respectively.

In the early 1980's Belavin and Drinfeld [1] classified the solutions to the MCYBE for the finite-dimensional complex simple Lie algebras and showed that the solution space $\mathcal{M}$ is a finite disjoint union of components of the projective space $\mathbb{P}(\mathfrak{g} \wedge \mathfrak{g})$. The components of $\mathcal{M}$ are indexed by triples $\mathcal{T} = (\mathcal{T}, \mathcal{S}_1, \mathcal{S}_2)$, where $\mathcal{S}_1$ and $\mathcal{S}_2$ are subsets of the set of simple roots of $\mathfrak{g}$ and $\mathcal{T} : \mathcal{S}_1 \to \mathcal{S}_2$ is a bijection that preserves the Killing form and satifies a nilpotency condition. For any BD-triple $(\mathcal{T}, \mathcal{S}_1, \mathcal{S}_2)$, one can always produce another BD-triple $\mathcal{T}'$ by restricting $\mathcal{T}$ to a subset of $\mathcal{S}_1$. This gives rise to the notion of a partial ordering on triples: $\mathcal{T}' < \mathcal{T}$. An interesting family of solutions of the MYCBE arises when considering maximal BD-triples with $\mathcal{S}_2$ missing a single root. These occur only in the case when $\mathfrak{g} = \mathfrak{sl}_n$, and in this setting there are exactly $\phi(n)$ BD-triples of this type, where ϕ is the Euler-totient function [8]. The component of $\mathcal{M}$ corresponding to such a triple can each be described as the SL_n-orbit of a single point $r \in \mathfrak{sl}_n \wedge \mathfrak{sl}_n$, called the *Cremmer–Gervais r-matrix* [4,7]. Hence, throughout we let i and n be a pair of positive coprime integers with $i < n$ and let $r_{CG}(i, n)$ denote the corresponding Cremmer–Gervais r-matrix of type (i, n). The BD-triple associated to $r_{CG}(i, n)$ has the i-th simple root missing from $\mathcal{S}_2$. The Cremmer–Gervais r-matrices have explicit formulas which we describe in Section 2.2. For example when $i = 1$ and $n = 3$, we have

$$\begin{aligned} r_{CG}(1,3) = 2e_{12} \wedge e_{32} + e_{12} \wedge e_{21} + e_{13} \wedge e_{31} + e_{23} \wedge e_{32} \\ + \frac{1}{3}(e_{11} - e_{22}) \wedge (e_{22} - e_{33}) \in \mathfrak{sl}_3 \wedge \mathfrak{sl}_3. \end{aligned}$$