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Fixed points and entropy of endomorphisms on simple abelian varieties

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ABSTRACT

In this paper we investigate fixed-point numbers and entropies of endomorphisms on abelian varieties. It was shown quite recently that the number of fixed-points of an iterated endomorphism on a simple complex torus is either periodic or grows exponentially. Criteria to decide whether a given endomorphism is of the one type or the other are still missing. Our first result provides such criteria for simple abelian varieties in terms of the possible types of endomorphism algebras. The number of fixed-points depends on the eigenvalues and we exactly show which analytic eigenvalues occur. This insight is also the starting point to ask for the entropy of an endomorphism. Our second result offers criteria for an endomorphism to be of zero or positive entropy. The entropy is computed as the logarithm of a real number and our third result characterizes the algebraic structure of this number.

Introduction

In the present paper we study the asymptotic behavior of the number of fixed-points of iterates of a holomorphic map of an abelian variety.

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The growth of the fixed-points function has been studied on two-dimensional complex tori in [2], where a complete classification for this case can be found. In that paper it was shown that the fixed-point function grows either exponentially or periodic or a combination of both. This result was recently extended to higher dimensions by Alvarado and Auffarth [1], who showed that the fixed-points function is still of one of these three types.

In this paper we extend the second theorem of [2], which deals with simple abelian surfaces, to higher dimensions. By [1] we know that endomorphisms on simple abelian varieties lead to a fixed-points function which grows either periodic or exponentially. Now it is desirable to know about the exact behavior of this function in terms of the possible types of endomorphism algebras.

Our first result contains this information for the cases that $\mathrm{End}_{\mathbb{Q}}(X)$ is a totally real number field, a totally definite quaternion algebra or a CM-field. We further classify the analytic eigenvalues with respect to the fixed-point behavior. In the case of multiplication by a totally indefinite quaternion algebra we show that, surprisingly, eigenvalues of absolute value 1 occur that are not roots of unity – this is in contrast with the surface case:

Theorem 1.

(1) *Let X be a simple abelian variety with $\mathrm{End}_{\mathbb{Q}}(X)$ isomorphic to a totally real number field, a definite quaternion algebra or a CM-field. Then for an endomorphism f we have:*
   - (a) *Suppose that X has real multiplication, i.e., $\mathrm{End}_{\mathbb{Q}}(X)$ is a totally real number field. Then $\#\,\mathrm{Fix}(f^n)$ is periodic if $f = \pm\,\mathrm{id}_X$, and it grows exponentially otherwise.*
   - (b) *Suppose that X has definite quaternion multiplication, i.e., $\mathrm{End}_{\mathbb{Q}}(X)$ is of the form $F + iF + jF + ijF$ with $i^2 = \alpha \in F \setminus \{0\}$, $j^2 = \beta \in F \setminus \{0\}$ and $ij = -ji$, where F is a totally real number field and $\mathrm{End}_{\mathbb{Q}}(X) \otimes_\sigma \mathbb{R} \simeq \mathbb{H}$ holds for every embedding $\sigma : F \hookrightarrow \mathbb{R}$. Write $f \in \mathrm{End}(X)$ as $f = a + bi + cj + dij$ with $a, b, c, d \in F$. Then $\#\,\mathrm{Fix}(f^n)$ is periodic if $|a + \sqrt{b^2\alpha + c^2\beta - d^2\alpha\beta}| = 1$, and it grows exponentially otherwise.*
   - (c) *Suppose that X has complex multiplication, i.e., $\mathrm{End}_{\mathbb{Q}}(X)$ is a CM-field. Then f has periodic fixed-point behavior if $|f| = 1$, and it has exponential fixed-points growth otherwise.*

   Moreover, if $\#\,\mathrm{Fix}(f^n)$ is periodic, then all analytic eigenvalues are roots of unity, and if $\#\,\mathrm{Fix}(f^n)$ grows exponentially, then they are all of absolute value $\neq 1$.

(2) *There exist simple abelian varieties with totally indefinite quaternion multiplication possessing endomorphisms with analytic eigenvalues of absolute value 1 which are not roots of unity.*

One can rephrase the first part of this theorem in the following way, using the Rosati involution: If $f \cdot f' = 1$ holds, then $\#\mathrm{Fix}(f^n)$ is periodic, and it grows exponentially otherwise.