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Mumford curves and Mumford groups in positive characteristic

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A Mumford group is a discontinuous subgroup Γ of $\mathrm{PGL}_2(K)$, where K denotes a non archimedean valued field, such that the quotient by Γ is a curve of genus 0. As abstract group Γ is an amalgam of a finite tree of finite groups. For K of positive characteristic the large collection of amalgams having two or three branch points is classified. Using these data Mumford curves with a large group of automorphisms are discovered. A long combinatorial proof, involving the classification of the finite simple groups, is needed for establishing an upper bound for the order of the group of automorphisms of a Mumford curve. Orbifolds in the category of rigid spaces are introduced. For the projective line the relations with Mumford groups and singular stratified bundles are studied. This paper is a sequel to [26]. Part of it clarifies, corrects and extends work of G. Cornelissen, F. Kato and K. Kontogeorgis.

Introduction

Let K be a complete non archimedean valued field. For convenience we will suppose that K is algebraically closed. A *Schottky group* Δ is a finitely generated, discontinuous

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subgroup of $\mathrm{PGL}_2(K)$ such that Δ contains no elements ($\neq 1$) of finite order and $\Delta \not\cong \{1\}, \mathbb{Z}$. It turns out that Δ is a free non-abelian group on $g > 1$ generators. Let $\Omega \subset \mathbb{P}^1_K$ denote the rigid open subspace of ordinary points for Δ. Then $X := \Omega/\Delta$ is an algebraic curve over K of genus g. The curves obtained in this way are called *Mumford curves*. Let $\Gamma \subset \mathrm{PGL}_2(K)$ denote the normalizer of Δ. Then Γ/Δ acts on X and is in fact the group of the automorphisms of X. For $K \supset \mathbb{Q}_p$, the theme of automorphisms of Mumford curves is of interest for p-adic orbifolds and for p-adic hypergeometric differential equations. According to F. Herrlich [15] one has for Mumford curves X of genus $g > 1$ the bound $|\mathrm{Aut}(X)| \leq 12(g-1)$ if $p > 5$. For $p = 2, 3, 5$ there are p-adic "triangle groups" and the bounds are $n_p(g-1)$ with $n_2 = 48,\ n_3 = 24,\ n_5 = 30$.

In this paper we investigate the case that K has characteristic $p > 0$.

The order of the automorphism group can be much larger than $12(g-1)$. Using the Riemann–Hurwitz–Zeuthen formula one easily shows (see also the proof of Corollary 6.2):

If $g > 1$ and $|\mathrm{Aut}(X)| > 12(g-1)$, then $X/\mathrm{Aut}(X) \cong \mathbb{P}^1_K$ and the morphism $X \to X/\mathrm{Aut}(X)$ is branched above 2 or 3 points.

There exist Mumford curves $X = \Omega/\Delta$ with genus $g > 1$ and such that $|\mathrm{Aut}(X)| > 12(g-1)$. Hence the normalizer Γ of $\Delta \subset \mathrm{PGL}(2, K)$ satisfies $\Omega/\Gamma \cong \mathbb{P}^1_K$. This leads to the definition of a *Mumford group*:

This is a finitely generated, discontinuous subgroup Γ of $\mathrm{PGL}_2(K)$ such that $\Omega/\Gamma \cong \mathbb{P}^1_K$, where $\Omega \subset \mathbb{P}^1_K$ is the rigid open subset of the ordinary points for the group Γ. We *exclude* the possibilities that Γ is finite and that Γ contains a subgroup of finite index, isomorphic to $\mathbb{Z}$. A point $a \in \mathbb{P}^1_K$ is called a *branch point* if a preimage $b \in \Omega$ of a has a non trivial stabilizer in Γ.

On the other hand, a Mumford group Γ contains a normal subgroup Δ, which is of finite index and has no elements $\neq 1$ of finite order. Thus Δ is a Schottky group, $X := \Omega/\Delta$ is a Mumford curve. Above we have excluded the cases that the genus of X is 0 or 1. The group $A := \Gamma/\Delta$ is a subgroup of $\mathrm{Aut}(X)$ such that $X/A \cong \mathbb{P}^1_K$.

In several papers [4–8,26,25] the construction and the classification of Mumford groups over a field K of characteristic $p > 0$ are studied. Here we continue this study. First we recall that a Mumford group is, as an abstract group, a finite tree of finite groups (T, G). In the work of F. Herrlich [15] and in [26] a criterion is proved which decides whether the 'amalgam' $\pi_1(T, G)$ of a finite tree of finite groups (T, G) is *realizable*, i.e., $\pi_1(T, G)$ has an embedding in $\mathrm{PGL}(2, K)$ as discontinuous group. If there is a realization, then, in general, there are some families of realizations. Thus in classifying Mumford groups we classify in fact the realizable finite trees of finite groups (T, G). Still, for the purpose of classification, there are too many realizable (T, G).

Since we are interested in Mumford curves X with $|\mathrm{Aut}(X)| > 12(g-1)$ and $g > 1$, we only consider trees of groups (T, G) which produce 2 or 3 branch points. The number of branch points br depends only on (T, G) and not on the chosen realization. A formula