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Universal Cartan–Lie algebroid of an anchored bundle with connection and compatible geometries

Alexei Kotov [a], Thomas Strobl [b,*]

[a] Faculty of Science, University of Hradec Kralove, Rokitanskeho 62, Hradec Kralove 50003, Czech Republic
[b] Institut Camille Jordan, Université Claude Bernard Lyon 1, Université de Lyon, 43 boulevard du 11 novembre 1918, 69622 Villeurbanne Cedex, France

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ABSTRACT

Consider an anchored bundle (E, ρ), i.e. a vector bundle $E \to M$ equipped with a bundle map $\rho : E \to TM$ covering the identity. M. Kapranov showed in the context of Lie–Rinehard algebras that there exists an extension of this anchored bundle to an infinite rank universal free Lie algebroid $FR(E) \supset E$. We adapt his construction to the case of an anchored bundle equipped with an arbitrary connection, (E, ∇), and show that it gives rise to a unique connection $\tilde{\nabla}$ on $FR(E)$ which is compatible with its Lie algebroid structure, thus turning $(FR(E), \tilde{\nabla})$ into a Cartan–Lie algebroid. Moreover, this construction is universal: any connection-preserving vector bundle morphism from (E, ∇) to a Cartan–Lie Algebroid $(A, \bar{\nabla})$ factors through a unique Cartan–Lie algebroid morphism from $(FR(E), \tilde{\nabla})$ to $(A, \bar{\nabla})$.

Suppose that, in addition, M is equipped with a geometrical structure defined by some tensor field t which is compatible with (E, ρ, ∇) in the sense of being annihilated by a natural E-connection that one can associate to these data. For example, for a Riemannian base (M, g) of an involutive anchored bundle (E, ρ), this condition implies that M carries a Riemannian foliation. It is shown that every E-compatible tensor field t becomes invariant with respect to the Lie algebroid representation associated canonically to the Cartan–Lie algebroid $(FR(E), \tilde{\nabla})$.

1. Introduction

Every vector space gives naturally rise to a free infinite-dimensional Lie algebra. Applying the same strategy to an anchored vector bundle

$$\begin{array}{ccc} E & \xrightarrow{\;\;\rho\;\;} & TM \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

needs some more care due to compatibility with the anchor map ρ, which, as a simple consequence of the Lie algebroid axioms, is required to become a morphism of the brackets. This implies in particular that in general the image of the anchor map will increase within this process. It is shown by M. Kapranov in [3] that any anchored module, a module over a commutative algebra together with a morphism of modules with values in the module of derivations of the algebra, gives

* Corresponding author.
E-mail addresses: oleksii.kotov@uhk.cz (A. Kotov), strobl@math.univ-lyon1.fr (T. Strobl).

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rise in a canonical way to a free infinite-dimensional Lie–Rinehart algebra.[1] A free Lie–Rinehart algebra admits a natural filtration the associated graded algebra to which is the free Lie algebra in the category of modules over the same algebra generated by this module. We apply the construction of Kapranov to the category of smooth real manifolds – the original paper operates with Lie–Rinehart algebras over arbitrary ground fields – and call the resulting Lie algebroid $FR(E) \to M$.

The main purpose of this article is, however, to extend this relation between an anchored bundle E and its free Lie algebroid $FR(E)$ to the lifting of particular additional structures from E to $FR(E)$ such that appropriate compatibility conditions are satisfied. For the case of a vector bundle connection, e.g., there is no natural compatibility condition to be required if the vector bundle is merely an anchored bundle; however, if it is a Lie algebroid, this changes: let $A \to M$ be a Lie algebroid and ∇ a connection on A. Any connection on A gives rise to a splitting ι_∇ of the natural projection map $J^1(A) \to A$, where $J^1(A)$ is the 1-jet bundle of sections of A. On the other hand, $J^1(A)$ carries a natural Lie algebroid structure itself, induced from the one on A. The compatibility consists of asking that $\iota_\nabla \colon A \to J^1(A)$ is a Lie algebroid morphism [1], in which case we call the connection a Cartan connection and the couple (A, ∇) a *Cartan–Lie algebroid*. This compatibility condition can be re-expressed [4] as the vanishing of the following tensor [5]

$$S := 2\mathrm{Alt}\langle \rho, F_\nabla \rangle + \nabla \left({}^{A}T\right), \tag{1}$$

where $F_\nabla \in \Gamma(A^* \otimes A \otimes \Lambda^2 T^*M)$ is the curvature of ∇, the anchor is considered as a section $\rho \in \Gamma(A^* \otimes TM)$, so that the contraction and skew-symmetrisation are defined in an obvious way, and ${}^{A}T$ is the A-torsion of the simple A-connection ${}^{A}\nabla$ on A defined by ${}^{A}\nabla_s(s') := \nabla_{\rho(s)}s'$ for all $s, s' \in \Gamma(A)$.

Theorem 1, proven in this paper, is a refinement of the above-mentioned result of Kapranov: given any anchored bundle E equipped with an arbitrary connection ∇ there is a unique Cartan connection $\tilde{\nabla}$ on the corresponding free Lie algebroid $FR(E)$ which extends the one on $E \subset FR(E)$. It is interesting to see that it is precisely the compatibility condition $S = 0$ which fixes the extension to all of $FR(E)$ uniquely. We call $(FR(E), \tilde{\nabla})$ the *free Cartan–Lie algebroid* generated by the anchored bundle with connection (E, ∇). $(FR(E), \tilde{\nabla})$ has a universality property, moreover, which we will specify further below. We mention as an aside that albeit we deal only with smooth manifolds in this paper, a purely algebraic version of this theorem in the spirit of [3] is quite obvious.

For an anchored bundle with connection (E, ∇) there is a natural compatibility with any tensor field t defined over its base M: define the E-connection ${}^{E}\nabla$ when acting on vector fields $v \in \Gamma(TM)$ by means of ${}^{E}\nabla_s v := [\rho(s), v] - \rho(\nabla_v s)$ for all $s \in \Gamma(E)$ and extend this canonically to all tensor fields over M. It is natural to ask that t should be annihilated by this E-derivative:

$${}^{E}\nabla t = 0\,. \tag{2}$$

The meaning of this condition becomes clearer with an example: suppose the image of the anchor map is involutive, $[\rho(\Gamma(E)), \rho(\Gamma(E))] \subset \rho(\Gamma(E))$, then $\rho(\Gamma(E))$ defines a singular foliation on M. If M is equipped with a metric g satisfying ${}^{E}\nabla g = 0$, then this singular foliation is Riemannian, and in particular transversally invariant with respect to the foliation. Similar statements hold true for other geometrical structures defined by means of a tensor field t satisfying Eq. (2). We note in parenthesis, if E carries in addition a Lie algebroid structure and the connection ∇ is compatible with it in the sense of $S = 0$, then ${}^{E}\nabla$ as defined above provides an honest Lie algebroid representation on TM, T^*M, and its tensor powers, and the compatibility with t then simply implies that this tensor is invariant under this canonical representation.[2]

The second result of the present paper, formulated in some generalisation in Proposition 1, is that geometrical structures t on M which are compatible with the anchored bundle with connection (E, ∇) in the sense of Eq. (2) remain compatible also with respect to $(FR(E), \tilde{\nabla})$, i.e. they become invariant with respect to the canonical representation of the universal free Cartan–Lie algebroid $(FR(E), \tilde{\nabla})$ on TM. Moreover, this construction is universal: *every* connection-preserving morphism from (E, ∇, t) to a t-compatible Cartan–Lie algebroid (A, ∇, t) such that the base map is the identity factors through this free Cartan Killing Lie algebroid.

Although it would be desirable to find conditions under which the free (Cartan–)Lie algebroid admits a finite-dimensional reduction, for the moment we leave this problem open. A necessary condition for such a reduction is that the – unmodified finite dimensional – base M of $FR(E)$ carries a singular foliation: in the infinite rank setting, involutivity of the image of $\rho_{FR(E)}$ is not sufficient for its integrability. In addition, even if $FR(E)$ admits a finite rank reduction, there are in general additional obstructions for the Cartan structure to reduce to the quotient Lie algebroid, since not every finite rank Lie algebroid even admits a compatible connection.

2. Anchored bundles and free Cartan–Lie algebroids

Let us denote by $Anch_c(M)$ the category whose objects are anchored bundles with connections and morphisms are connection-preserving bundle morphisms commuting with the anchor maps. Let $CLie(M)$ be the category of Cartan–Lie algebroids over M. Every Cartan–Lie algebroid is an anchored bundle and every connection-preserving Lie algebroid morphism is a morphism of the underlying anchored bundle structures, thus there is a natural forgetful functor

$$CLie(M) \to Anch_c(M)\,. \tag{3}$$

[1] A Lie–Rinehart algebra is an algebraic counterpart of a Lie algebroid, cf. [7].

[2] We refer the reader to [4] for proofs and further details about the statements in this paragraph.