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Weyl and Browder S-spectra in a right quaternionic Hilbert space

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ABSTRACT

In this note first we study the Weyl operators and Weyl S-spectrum of a bounded right quaternionic linear operator, in the setting of the so-called S-spectrum, in a right quaternionic Hilbert space. In particular, we give a characterization for the S-spectrum in terms of the Weyl operators. In the same space we also study the Browder operators and introduce the Browder S-spectrum.

1. Introduction

In the complex theory the concept of Weyl spectrum and Browder spectrum are subjects of the theory of perturbation of the spectrum, however it has found applications in operator theory and related areas [13,9,14]. In the complex case, the Weyl spectrum of a bounded linear operator is the largest part of the spectrum that is invariant under compact perturbations [14,9]. We shall show that the same is true in the quaternionic Weyl S-spectrum. However, in the complex case, the Browder spectrum is not invariant under compact perturbations [14].

In the complex setting, in a Hilbert space $\mathfrak{H}$, for a bounded linear operator, A, the point spectrum or the eigenvalues of A contain isolated eigenvalues of finite algebraic and geometric multiplicities. Also these sets are important in the study of Weyl and Browder spectra [14]. In the quaternionic setting, let $V_{\mathbb{H}}^{R}$ be a separable right Hilbert space, A be a bounded right linear operator, and $R_{\mathfrak{q}}(A) = A^2 - 2\text{Re}(\mathfrak{q})A + |\mathfrak{q}|^2\mathbb{I}_{V_{\mathbb{H}}^{R}}$, with $\mathfrak{q} \in \mathbb{H}$, the set of all quaternions and $\mathbb{I}_{V_{\mathbb{H}}^{R}}$ be the identity operator on $V_{\mathbb{H}}^{R}$, be the pseudo-resolvent operator, the set of right eigenvalues of $R_{\mathfrak{q}}(A)$ coincides with the point S-spectrum (see proposition 4.5 in [11]). In this regard, it will be appropriate to define and study the quaternionic isolated S-point spectrum as the quaternions which are eigenvalues of $R_{\mathfrak{q}}(A)$.

Due to the non-commutativity, in the quaternionic case there are three types of Hilbert spaces: left, right, and two-sided, depending on how vectors are multiplied by scalars. This fact can entail several problems. For example, when a Hilbert space

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$\mathcal{H}$ is one-sided (either left or right) the set of linear operators acting on it does not have a linear structure. Moreover, in a one sided quaternionic Hilbert space, given a linear operator A and a quaternion $\mathfrak{q} \in \mathbb{H}$, in general we have that $(\mathfrak{q}A)^\dagger \neq \overline{\mathfrak{q}}A^\dagger$ (see [16] for details). These restrictions can severely prevent the generalization to the quaternionic case of results valid in the complex setting. Even though most of the linear spaces are one-sided, it is possible to introduce a notion of multiplication on both sides by fixing an arbitrary Hilbert basis of $\mathcal{H}$. This fact allows to have a linear structure on the set of linear operators, which is a minimal requirement to develop a full theory. Thus, the framework of this paper, is in part, is a right quaternionic Hilbert space equipped with a left multiplication, introduced by fixing a Hilbert basis.

In the study of Weyl and Browder S-spectra, the essential S-spectra gets involved. In defining the essential S-spectrum the structure of the so-called quaternionic Calkin algebra is used, in which the set of all bounded quaternionic right linear operators, $\mathcal{B}(V_{\mathbb{H}}^R)$, should form a quaternionic two-sided Banach C^*-algebra with unity. This can only happen if we consider $V_{\mathbb{H}}^R$ with a left multiplication defined on it, which is a basis dependent multiplication [11]. However, regardless of which basis we choose the set $\mathcal{B}(V_{\mathbb{H}}^R)$ will become a quaternionic two-sided Banach C^*-algebra with unity. Thus, the invariance under a basis change naturally exists.

As far as we know, Weyl and Browder operators and the Weyl and Browder S-spectra have not been studied in the quaternionic setting yet. In this regard, in this note we investigate the quaternionic Weyl operators and Weyl S-spectrum and provide a characterization to the S-spectrum in terms of the Weyl operators (see Theorem 6.6). We also study the Browder operators to certain extent and introduce the Browder spectrum. However, in the complex case, the Browder spectrum and its characterizations depend on the so-called Riesz idempotent which is defined in terms of the Cauchy integral formula for operators [14]. In the quaternionic setting, the Cauchy integral formula, and thereby the S-functional calculus, is known only for the slice regular functions and it is defined on an axially symmetric domain in quaternion slices. A quaternion slice is a complex plane contained in the set of all quaternions [12,4,5]. In this regard, this fact severely affected our ability in studying the Browder S-spectrum in broad on the whole set of quaternions. However, one may be able to study it in an axially symmetric domain. Also in the study of quaternionic Weyl and Browder operators and S-spectra results regarding quaternionic Fredholm operators and quaternionic essential S-spectrum are involved. Materials regarding these two topics are heavily borrowed from the recent paper [15] as needed here.

The article is organized as follows. In Section 2 we introduce the set of quaternions and quaternionic Hilbert spaces and their bases, as needed for the development of this article, which may not be familiar to a broad range of audience. In Section 3 we define and investigate, as needed, right linear operators and their properties. In Section 3.1 we define a basis dependent left multiplication on a right quaternionic Hilbert space. In Section 3.2 we deal with the right S-spectrum, left S-spectrum, S-spectrum and its major partitions. In Section 4 we recall some facts about the Fredholm operators and its index for a bounded quaternionic right linear operator from [15]. In Section 5, from [15] we recall results about the essential S-spectrum as needed. We also prove certain results which are omitted from [15]. In Section 6 we introduce quaternionic Weyl operators and Weyl S-spectrum. In particular we provide a characterization to the S-spectrum in terms of the quaternionic Weyl operators. In Section 7 we define and study the quaternionic Browder operators and Browder S-spectrum in a limited sense, which is due to the unavailability of a Cauchy integral formula on the whole set of quaternions. Section 8 ends the manuscript with a brief conclusion.

2. Mathematical preliminaries

In order to make the paper self-contained, we recall some facts about quaternions which may not be well-known. For details we refer the reader to [1,11,18].

2.1. Quaternions

Let $\mathbb{H}$ denote the field of all quaternions and $\mathbb{H}^*$ the group (under quaternionic multiplication) of all invertible quaternions. A general quaternion can be written as

$$\mathfrak{q} = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}, \qquad q_0, q_1, q_2, q_3 \in \mathbb{R},$$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the three quaternionic imaginary units, satisfying $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and $\mathbf{ij} = \mathbf{k} = -\mathbf{ji}$, $\mathbf{jk} = \mathbf{i} = -\mathbf{kj}$, $\mathbf{ki} = \mathbf{j} = -\mathbf{ik}$. The quaternionic conjugate of $\mathfrak{q}$ is

$$\overline{\mathfrak{q}} = q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3,$$

while $|\mathfrak{q}| = (\mathfrak{q}\overline{\mathfrak{q}})^{1/2}$ denotes the usual norm of the quaternion $\mathfrak{q}$. If $\mathfrak{q}$ is non-zero element, it has inverse $\mathfrak{q}^{-1} = \frac{\overline{\mathfrak{q}}}{|\mathfrak{q}|^2}$. Finally, the set

$$\mathbb{S} = \{I = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \mid x_1, x_2, x_3 \in \mathbb{R},\ x_1^2 + x_2^2 + x_3^2 = 1\},$$

contains all the elements whose square is -1. It is a 2-dimensional sphere in $\mathbb{H}$ identified with $\mathbb{R}^4$.

2.2. Quaternionic Hilbert spaces

In this subsection we discuss right quaternionic Hilbert spaces. For more details we refer the reader to [1,11,18].