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The Frobenius–Virasoro algebra and Euler equations-II: Multi-component cases

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ABSTRACT

Let $W_{\mathcal{A}}$ be the vector field Lie algebra with values in the Frobneius algebra $\mathcal{A}$ of rank N. We show that the second continuous cohomology group is

$$\mathrm{H}^2(W_{\mathcal{A}}, \mathbb{K}) = \bigoplus_{k=1}^{N} \mathbb{K}\,\omega_k, \quad \mathbb{K} = \mathbb{C} \quad \text{or} \quad \mathcal{A},$$

where $\omega_1, \ldots, \omega_N$ are $\mathbb{C}$-valued basic Gelfand–Fuchs cocycles. As an application, we present a generalized Virasoro algebra $\mathfrak{vir}_{\mathcal{A}[\alpha_1,\ldots,\alpha_k]}$. We also consider another kind of generalization and propose a multi-component Frobenius–Virasoro algebra $\mathfrak{vir}_{\mathcal{A}}^{[n]}$ and study its related Euler-type equations.

1. Introduction

Let $\mathcal{A}$ be a Frobenius algebra over $\mathbb{C}$ with the basis $\{e^1, \ldots, e^N\}$ satisfying $e^i \circ e^j = \sum_{k=1}^{N} C_k^{ij} e^k$. Let

$$W_{\mathcal{A}} = \left\{ U(x)\partial \,|\, U \in C^{\infty}(\mathbb{S}^1, \mathcal{A}) \right\}, \quad \partial = \frac{d}{dx} \tag{1.1}$$

be the set of all vector fields valued in $\mathcal{A}$ with the commutation relation

$$[U\partial, V\partial] = (U \circ V_x - V \circ U_x)\partial, \quad U, V \in C^{\infty}(\mathbb{S}^1, \mathcal{A}). \tag{1.2}$$

Then $(W_{\mathcal{A}}, [\ ,\ ])$ forms a Lie algebra [4,26], called to be the Frobenius–Witt algebra. For any $U(x)\partial \in W_{\mathcal{A}}$, it could be expanded into an $\mathcal{A}$-valued Fourier series

$$U(x)\partial = -\sqrt{-1} \sum_{k=1}^{N} \sum_{m \in \mathbb{Z}} u_{k,m} L_m^k, \quad u_{k,m} \in \mathbb{C},$$

where $L_m^k = \sqrt{-1} \exp\left(\sqrt{-1} m x\right) e^k \partial$, then

$$[L_m^i, L_n^j] = (m - n) \sum_{k=1}^{N} C_k^{ij} L_{m+n}^k. \tag{1.3}$$

So, $\{L_m^k \,|\, m \in \mathbb{Z},\ k = 1, \ldots, N\}$ could be regarded as a basis for the Frobenius–Witt algebra $W_{\mathcal{A}}$.

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When $\mathcal{A}=\mathbb{C}$, the Frobenius–Witt algebra $W_{\mathbb{C}}$ is the Witt algebra, which has a unique central extension called the Virasoro algebra $\mathfrak{vir}=W_{\mathbb{C}}\bigoplus\mathbb{C}Z$. More precisely, write $L_m=L^1_m$,

$$[L_m,L_n]=(m-n)L_{m+n}+\delta_{m+n,0}\frac{m^3-m}{12}Z,\quad [Z,L_m]=0,\quad m,\ n\in\mathbb{Z}.$$

The two-cocycle

$$\omega_{\mathbb{C}}(L_m,L_n)=\delta_{m+n,0}\frac{m^3-m}{12}\in H^2(W_{\mathbb{C}},\mathbb{C})=\mathbb{C}$$

is called the Gelfand–Fuchs cocycle [11,14,19]. The Virasoro algebra is closely related to Kac–Moody algebras and plays an important role in 2D conformal field theory, please see e.g. [9,12] and references therein.

A natural question is to study possible central extensions of the Frobenius–Witt algebra $W_{\mathcal{A}}$. For instance, in [3,4], they proposed a one-dimensional central extension as follows

$$[L^i_m,L^j_n]=(m-n)\sum_{k=1}^{N}C^{ij}_kL^k_{m+n}+\sum_{k=1}^{N}\alpha_kC^{ij}_k\frac{m^3-m}{12}\delta_{m+n,0}Z,\tag{1.4}$$

where Z is the central element and $\alpha_k\in\mathbb{C}$. In [26], with the help of the $\mathcal{A}$-valued Gelfand–Fuchs cocycle

$$\omega_{\mathcal{A}}(U\partial,V\partial)=\int_{\mathbb{S}^1}U\circ V_{xxx}dx,\tag{1.5}$$

we introduce the Frobenius–Virasoro algebra $\mathfrak{vir}^{[0]}_{\mathcal{A}}=W_{\mathcal{A}}\oplus\mathcal{A}$ as an N-dimensional central extension of $W_{\mathcal{A}}$. More precisely,

$$[L^i_m,L^j_n]=(m-n)\sum_{k=1}^{N}C^{ij}_kL^k_{m+n}+\sum_{k=1}^{N}C^{ij}_k\frac{m^3-m}{12}\delta_{m+n,0}Z^k,\tag{1.6}$$

where Z^k are central elements for $k=1,\ldots,N$. Especially when $\mathcal{A}=\mathcal{F}^0_2$, this is the so-called $W(2,2)$-algebra [2,10,23]. We remark that $\mathrm{H}^2(W_{\mathcal{A}},\mathcal{A})$ if $N>1$ is not generated by the $\mathcal{A}$-valued Gelfand–Fuchs cocycle $\omega_{\mathcal{A}}$. An interesting problem is to compute $\mathrm{H}^2(W_{\mathcal{A}},\mathcal{A})$.

This paper is organized as follows. In Section 2, we will study all possible central extensions of the Frobenius–Witt algebra $W_{\mathcal{A}}$ and show that

$$\mathrm{H}^2(W_{\mathcal{A}},\mathbb{K})=\bigoplus_{k=1}^{N}\mathbb{K}\,\omega_k,\quad \mathbb{K}=\mathbb{C}\quad\text{or}\quad\mathcal{A},$$

where $\omega_1,\ldots,\omega_N$ are $\mathbb{C}$-valued basic Gelfand–Fuchs cocycles (see below (2.3)). As a corollary, we obtain a generalized Virasoro algebra

$$\mathfrak{vir}_{\mathcal{A}[\alpha_1,\ldots,\alpha_k]}=\bigoplus_{m\in\mathbb{Z}}\bigoplus_{i=1}^{N}\mathbb{C}L^i_m\bigoplus_{k=1}^{N}\mathbb{C}Z^k$$

equipped with the commutation relation:

$$[L^i_m,L^j_n]=(m-n)\sum_{k=1}^{N}C^{ij}_kL^k_{m+n}+\sum_{k=1}^{N}\alpha_kC^{ij}_k\frac{m^3-m}{12}\delta_{m+n,0}Z^k,\tag{1.7}$$

where Z^k are central elements and $\alpha_k\in\mathbb{C}$ for $k=1,\ldots,N$. In Section 3, we will propose a multi-component Frobenius–Virasoro algebra $\mathfrak{vir}^{[n]}_{\mathcal{A}}$ as another generalization of $\mathfrak{vir}^{[0]}_{\mathcal{A}}$ and study its related Euler equations including the $\mathcal{A}$-valued multi-component KdV equation, the $\mathcal{A}$-valued multi-component Camassa–Holm equation and the $\mathcal{A}$-valued multi-component Hunter–Saxton equation.

2. Central extensions of the Frobenius–Witt algebra $W_{\mathcal{A}}$

In this section, we will study all possible central extensions of the Frobenius–Witt algebra $W_{\mathcal{A}}$ defined in (1.1). Let us begin with some definitions.

Definition 2.1. A Frobenius algebra $\mathcal{A}:=\{\mathcal{A},\mathrm{tr}_{\mathcal{A}},\mathbf{1}_{\mathcal{A}},\circ\}$ over $\mathbb{C}$ is a free $\mathbb{C}$-module $\mathcal{A}$ of a finite rank N, equipped with a commutative[1] and associative multiplication $\circ$ and a unit $\mathbf{1}_{\mathcal{A}}$, and a $\mathbb{C}$-linear form $\mathrm{tr}_{\mathcal{A}}:\mathcal{A}\to\mathbb{C}$ whose kernel contains no nontrivial ideas.

1 Generally (e.g.,[17]), the Frobenius algebra does not require the commutativity. For our purpose, we always assume that it is commutative [21,22,26].