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Differential forms with values in VB-groupoids and its Morita invariance

Thiago Drummond [a,*], Leandro Egea [b]

[a] Universidade Federal do Rio de Janeiro, Instituto de Matemática, Caixa Postal 68530, Rio de Janeiro, RJ, 21941-909, Brazil
[b] Universidade Federal Fluminense, Departamento de Matemática, Rua Desembargador Ellis Hermydio Figueira, 783 Aterrado, Volta Redonda RJ 27213-415, Brazil

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ABSTRACT

We introduce multiplicative differential forms on Lie groupoids with values in VB-groupoids. Our main result gives a complete description of these objects in terms of infinitesimal data. By considering split VB-groupoids, we are able to present a Lie theory for differential forms on Lie groupoids with values in 2-term representations up to homotopy. We also define a differential complex whose 1-cocycles are exactly the multiplicative forms with values in VB-groupoids and study the Morita invariance of its cohomology.

1. Introduction

This paper is devoted to the study of differential forms on Lie groupoids with coefficients. While multiplicative differential forms with values in representations of Lie groupoids have been treated in the literature, see e.g. [12], here we consider the broader context of forms with values in VB-groupoids. Recall that VB-groupoids are, roughly, vector bundles in the category of Lie groupoids. They naturally extend the notion of Lie-groupoid representations by encoding the information of representations *up to homotopy* [19] (see also [2]), with the tangent bundle playing the role of the adjoint representation. The infinitesimal counterparts of VB-groupoids are known as VB-algebroids. In this paper, we introduce the notion of multiplicative differential form with values in VB-groupoids and establish two main results: first, we provide a purely infinitesimal description of these objects, extending the works in [3,6,12]; second, we describe a cohomology theory for such differential forms, that we prove to be Morita invariant.

Multiplicative differential forms (with trivial coefficients) on Lie groupoids appear in various contexts, often in connection with the (Lie-theoretic) integration of geometric structures: e.g. in symplectic and pre-symplectic groupoids [8,33], which are the global objects integrating Poisson and Dirac structures, respectively. The Lie theory of multiplicative differential forms with trivial coefficients is by now totally understood [3,6]. More recently, differential forms with coefficients in a representation and their infinitesimal versions, known as Spencer operators, were studied in [12] in an effort to understand the work of Cartan on Lie pseudogroups from a global perspective (see also [13]). One important application of [12] is providing the infinitesimal characterization of certain multiplicative distributions on Lie groupoids, a relevant topic due to its relation with quantization of Poisson manifolds [21] and the Lie theory of Dirac structures [24,25]. There is, however, an additional requirement the distributions studied in [12] must satisfy: the tangent space of the manifold of units of the Lie groupoid must be contained in the distribution. By allowing more general VB-groupoids as coefficients, we are able to drop this extra condition and develop a Lie theory for general multiplicative distributions.

* Corresponding author.
E-mail addresses: drummond@im.ufrj.br (T. Drummond), lgegea@icmc.usp.br (L. Egea).

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There is yet another important feature of differential forms on Lie groupoids: their multiplicativity is a cocycle condition on a differential complex, known as the Bott–Shulman complex. Its cohomology is a Morita invariant of the Lie groupoid related to the de Rham cohomology of the associated classifying space [4]. Here, we introduce a differential complex for forms with coefficients generalizing the Bott–Shulman complex and prove the Morita invariance of its cohomology. In view of the recent work on the Morita invariance of VB-groupoids [23], it is to be expected that this complex will be a useful tool in understanding connections on vector bundles over differentiable stacks.

Statement of results. To explain our main results it is necessary to recall some facts regarding VB-groupoids. These are presented in greater detail in Section 2.

Let $\mathcal{G} \rightrightarrows M$ be a Lie groupoid and $\mathcal{V} \rightrightarrows E$ be a VB-groupoid over $\mathcal{G}$. We denote by $C \to M$ the *core bundle of* $\mathcal{V}$; it is defined as the kernel of the source map (seen as a vector bundle morphism) $\widetilde{\mathtt{s}} : \mathcal{V}|_M \to E$. The target map $\widetilde{\mathtt{t}} : \mathcal{V}|_M \to E$ induces a vector bundle morphism $\partial : C \to E$ known as the *core–anchor*.

The Lie algebroid of $\mathcal{G}$ and the VB-algebroid of V are denoted by $A \to M$ and $\mathfrak{v} \to E$, respectively. Recall that $\mathfrak{v}$ is also a double vector bundle, where the second vector bundle structure $\mathfrak{v} \to A$ comes from applying the Lie functor to the structure maps of the vector bundle $\mathcal{V} \to \mathcal{G}$. The sections of $\mathfrak{v} \to E$ which are vector bundle morphisms from $E \to M$ to $\mathfrak{v} \to A$ are known as *linear sections* and we denote them as $\Gamma_{lin}(E, \mathfrak{v})$. The projection of a linear section on the section of A it covers is denoted by $\mathrm{pr} : \Gamma_{lin}(E, \mathfrak{v}) \to \Gamma(A)$.

A fundamental result regarding linear sections is that the right-invariant vector fields of $\mathcal{V}$ coming from $\Gamma_{lin}(E, \mathfrak{v})$ are linear [16, Prop. 3.9] (see also Proposition 2.7). So, there is a derivation $\Delta_\eta : \Gamma(\mathcal{V}) \to \Gamma(\mathcal{V})$ and a Lie derivative operator $L_{\Delta_\eta} : \Omega(\mathcal{G}, \mathcal{V}) \to \Omega(\mathcal{G}, \mathcal{V})$ corresponding to each element $\eta \in \Gamma_{lin}(E, \mathfrak{v})$. We review basic facts about linear vector fields and derivations on vector bundles in the Appendix . Recently, in [16], derivations of VB-groupoids coming from multiplicative linear vector fields on $\mathcal{V}$ were studied. It is important to stress that their derivations are not related to ours.

We are now able to state our main result in the Lie theory of differential forms. First, an element $\vartheta \in \Omega^q(\mathcal{G}, \mathcal{V})$ is said to be *multiplicative* if it defines a VB-groupoid morphism when seen as a map $T\mathcal{G} \oplus \cdots \oplus T\mathcal{G} \to \mathcal{V}$.

Theorem 1.1. *If $\mathcal{G} \rightrightarrows M$ is a source 1-connected groupoid, then there is a natural 1–1 correspondence between multiplicative forms $\vartheta \in \Omega^q(\mathcal{G}, \mathcal{V})$ and triples (D, l, θ), where $l : A \to \wedge^{q-1}T^*M \otimes C$ is a vector bundle map, $\theta \in \Omega^q(M, E)$ and $D : \Gamma_{lin}(E, \mathfrak{v}) \to \Omega^q(M, C)$ satisfies*

$$D(\mathcal{B}\Phi) = -\Phi \circ \theta, \quad D(f\eta) = fD(\eta) + df \wedge l(\mathrm{pr}(\eta)), \tag{1.1}$$

where $f \in C^\infty(M)$, $\eta \in \Gamma_{lin}(E, \mathfrak{v})$, $\mathcal{B}\Phi \in \Gamma_{lin}(E, \mathfrak{v})$ is the linear section covering the zero section of A corresponding to a vector bundle morphism $\Phi : E \to C$. Also, the following equations hold, for $\eta_1, \eta_2 \in \Gamma_{lin}(E, \mathfrak{v})$, $\alpha, \beta \in \Gamma(A)$:

$$D([\eta_1, \eta_2]) = L_{\nabla_{\eta_1}} D(\eta_2) - L_{\nabla_{\eta_2}} D(\eta_1) \tag{IM1}$$

$$l([\mathrm{pr}(\eta), \beta]) = L_{\nabla_\eta} l(\beta) - i_{\rho(\beta)} D(\eta) \tag{IM2}$$

$$i_{\rho(\alpha)} l(\beta) = -i_{\rho(\beta)} l(\alpha) \tag{IM3}$$

$$L_{\nabla_\eta} \theta = \partial(D(\eta)) \tag{IM4}$$

$$i_{\rho(\alpha)} \theta = \partial(l(\alpha)), \tag{IM5}$$

where ∇ is the fat representation of $\Gamma_{lin}(E, \mathfrak{v})$ on $\partial : C \to E$, $\rho : A \to TM$ is the anchor of A. The triple (D, l, θ) is obtained from ϑ by the formulas:

$$D(\eta) = L_{\Delta_\eta}(\vartheta)|_M, \quad l(\alpha) = i_{\overrightarrow{\alpha}} \vartheta|_M, \quad \theta = \vartheta|_M.$$

We shall refer to the set of Eqs. (IM1)–(IM5) as the *IM equations* (where IM stands for "infinitesimally multiplicative") and to a triple (D, l, θ) satisfying them together with (1.1) as an *IM q-form on A with values in* $\mathfrak{v}$.

In Section 3, we show how Theorem 1.1 recovers previous infinitesimal-global results in the literature regarding differential forms on Lie groupoids. In this section, we also show how multiplicative forms with values in VB-groupoids and IM forms with values in VB-algebroids give rise to a notion of multiplicative forms and IM forms with values in representations up to homotopy. Some of the equations appearing in this context have also appeared in [32] in connection with higher gauge theory.

A distribution $\mathcal{H} \subset T\mathcal{G}$ on the Lie groupoid $\mathcal{G}$ is called *multiplicative* if it is a subgroupoid of the tangent groupoid $T\mathcal{G}$. These distributions can be studied via Theorem 1.1 by considering the quotient projection $\vartheta : T\mathcal{G} \to T\mathcal{G}/\mathcal{H}$ as a 1-form with value in a VB-groupoid. In Section 6, we show how the resulting IM 1-form on A with values in $\mathrm{Lie}(T\mathcal{G}/\mathcal{H})$ can be refined to give an infinitesimal description of multiplicative distributions using the notion of *IM distributions*. The infinitesimal-global correspondence between multiplicative and IM distributions (Theorem 6.13) recovers the result of [12] characterizing multiplicative distributions when the base manifold of $\mathcal{H}$ is TM. It is important to mention that the set of equations satisfied by IM distributions does not depend on the choice of connections on A. This improves the characterization of VB-subalgebroids of TA appearing in [14].

Our approach to prove Theorem 1.1 is built on ideas developed in [6,9]. Namely, the multiplicativity of $\vartheta \in \Omega^q(\mathcal{G}, \mathcal{V})$ can be characterized by a cocycle equation for the corresponding function on the big groupoid

$$\mathbb{G} = T\mathcal{G} \oplus \cdots \oplus T\mathcal{G} \oplus \mathcal{V}^*$$