European Journal of Combinatorics 76 (2019) 27–36

Contents lists available at ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Cuts in matchings of 3-connected cubic graphs

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ARTICLE INFO

Article history:
Received 17 December 2017
Accepted 7 September 2018

ABSTRACT

We discuss conjectures on Hamiltonicity in cubic graphs (Tait, Barnette, Tutte), on the dichromatic number of planar oriented graphs (Neumann-Lara), and on even graphs in digraphs whose contraction is strongly connected (Hochstättler). We show that all of them fit into the same framework related to cuts in matchings. This allows us to find a counterexample to the conjecture of Hochstättler and show that the conjecture of Neumann-Lara holds for all planar graphs on at most 26 vertices. Finally, we state a new conjecture on bipartite cubic oriented graphs, that naturally arises in this setting.

1. Introduction

Let us first introduce three conjectures on 3-connected, cubic graphs in their order of appearance.

Conjecture 1 (*[16]*)**.** *Every 3-connected, cubic, planar graph contains a Hamiltonian cycle.*

The first counterexample to Conjecture 1 was given by Tutte [18] and is a graph on 46 vertices. Several smaller counterexamples on 38 vertices were later found by Holton and McKay, who also proved that there is no counterexample with less than 38 vertices [9]. One can observe that all known counterexamples to Tait's conjecture have odd cycles. Maybe this is essential:

Conjecture 2 (*Barnette 1969 [2]*)**.** *Every 3-connected, cubic, planar, bipartite graph contains a Hamiltonian cycle.*

In general Conjecture 2 remains open. It was shown to be true for graphs with at most 66 vertices [8]. There is an announcement [1] claiming that the conjecture was verified for graphs with less than 86 vertices. A few years later a stronger conjecture was proposed:

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https://doi.org/10.1016/j.ejc.2018.09.004

Conjecture 3 (*Tutte 1971 [19]*)**.** *Every 3-connected, cubic, bipartite graph contains a Hamiltonian cycle.*

Conjecture 3 was disproved by Horton [10]. The smallest known counterexample has 50 vertices and was discovered independently by Georges [5] and Kelmans [11]. Moreover, in [11] it is claimed (but no reference given) that Lomonosov and Kelmans proved Conjecture 3 for graphs on at most 30 vertices. We verified Conjecture 3 on graphs up to 40 vertices by computer.

Let us now go back to the plane and introduce a seemingly unrelated conjecture in digraphs. The *digirth* of a digraph $D = (V, A)$ is the length of a shortest directed cycle. A digraph is called *oriented graph* if it is of digirth at least 3. A set of vertices $V' \subseteq V$ is acyclic in D if the digraph induced by V' contains no directed cycle. Neumann-Lara stated the following:

Conjecture 4 (*Neumann-Lara 1985 [15]*)**.** *Every planar oriented graph can be vertex-partitioned into two acyclic sets.*

Conjecture 4 remains open in general, but was recently proved for graphs with digirth at least 4 [12]. Here we give the first computational evidence for it by showing that it is valid for all planar graphs on at most 26 vertices, see Proposition 4.

Let us introduce another seemingly unrelated problem. Given a (partially) directed graph $D = (V, A)$, for $E \subseteq A$, let D/E denote the graph obtained from D by contracting the edges of E. An *even* subgraph E of a digraph $D = (V, A)$ is a subset $E \subseteq A$ that is an edge-disjoint union of cycles of D (the cycles are not necessarily directed). Recently, Hochstättler proposed:

Conjecture 5 (*Hochstättler 2017 [7]*)**.** *In every 3-edge-connected digraph $D = (V, A)$ there exists an even subgraph $E \subseteq A$ such that D/E is strongly connected.*

We construct a counterexample to Conjecture 5 on 122 vertices, see Proposition 3.

In Section 2 we explain that all these conjectures arise in the same context and naturally lead to a new question (Conjecture 6). In Section 3 we give a general method that might be helpful to search for counterexamples to Conjectures 4 and 6. In particular, this method leads to a counterexample to Conjecture 5. We explain our computational results in Section 4 and conclude the paper in Section 5.

2. Cubic graphs without perfect matchings containing a cut

In the present section we will reformulate all of the above conjectures as statements about cubic (planar, bipartite, directed) graphs with some perfect matchings not containing a (directed) cut. See Fig. 1 for an illustration of the relations between the conjectures discussed in this section.

First, observe that a cubic graph $G = (V, E)$ contains a Hamiltonian cycle C, if and only if the complement $E \setminus C$ is a perfect matching containing no edge-cut. Therefore, in Conjectures 1, 2, 3 the property *contains a Hamiltonian cycle* can be replaced equivalently by *contains a perfect matching without cut*. This section is dedicated to the proof that Conjectures 4 and 5 can be reformulated equivalently as below, which justifies their placement in Fig. 1.

Conjecture IV (*Neumann-Lara*)**.** *Every 3-connected, cubic, planar digraph contains a perfect matching without a directed cut.*

Conjecture V (*Hochstättler*)**.** *Every 3-edge-connected, cubic digraph contains a perfect matching without a directed cut.*

Before proceeding to the proof, note that a new question follows naturally:

Conjecture 6. *Every 3-connected, cubic, bipartite digraph contains a perfect matching without directed cut.*

Let us now proceed to the proof, which we have split into several lemmas.

Lemma 1. *Conjecture 4 is equivalent to Conjecture 5 restricted to planar graphs.*