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Combinatorial proofs and generalizations of conjectures related to Euler's partition theorem

Jane Y.X. Yang

School of Science, Chongqing University of Posts and Telecommunications, Chongqing 400065, PR China

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ABSTRACT

In a recent work, Andrews gave analytic proofs of two conjectures concerning some variations of two combinatorial identities between partitions of a positive integer into odd parts and partitions into distinct parts discovered by Beck. Subsequently, using the same method as Andrews, Chern presented the analytic proof of another Beck's conjecture relating the gap-free partitions and distinct partitions with odd length. However, the combinatorial interpretations of these conjectures are still unclear and required. In this paper, motivated by Glaisher's bijection, we give the combinatorial proofs of these three conjectures directly or by proving more generalized results.

1. Introduction

A *partition* [1] of n is a finite nonincreasing sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that $n = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$. We write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ and call λ_i's the *parts* of λ. If a part i has multiplicity m_i for $i \geq 1$, we also write λ as $(1^{m_1}, 2^{m_2}, \ldots)$, where the superscript m_i is neglected provided $m_i = 1$. The *weight* of λ is the sum of all parts, which is denoted by $|\lambda|$, and the *length* of λ is the number of parts, which is denoted by $\ell(\lambda)$. The *conjugate* of λ is the partition $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_{\lambda_1})$, where $\lambda'_i = |\{\lambda_j : \lambda_j \geq i, 1 \leq j \leq \ell\}|$ for $1 \leq i \leq \lambda_1$, or λ' can be equivalently expressed as $(1^{\lambda_1-\lambda_2}, 2^{\lambda_2-\lambda_3}, \ldots, \ell-1^{\lambda_{\ell-1}-\lambda_\ell}, \ell^{\lambda_\ell})$. A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is called a distinct partition if $\lambda_1 > \lambda_2 > \cdots > \lambda_\ell$, and an odd partition if λ_i is odd for all $1 \leq i \leq \ell$, respectively. In 1748, by using generating functions, Euler [6] gave the celebrated partition theorem as follows.

Theorem 1.1 (*Euler's Partition Theorem*)**.** *The number of distinct partitions of n equals the number of odd partitions of n.*

E-mail address: yangyingxue@cqupt.edu.cn.

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After Euler's partition theorem was proposed, there have been many extensions and refinements, the famous ones of which are Glaisher's theorem and Franklin's theorem, and the reader can refer to [11,12,16] for more details. Glaisher [10] bijectively proved the following extension.

Theorem 1.2 (*Glaisher's Theorem*). *For any positive integer k, the number of partitions of n with no part occurring k or more times equals the number of partitions of n with no part divisible by k.*

In 1882, Franklin [8,13] acquired a more generalized result by giving constructive proof of the following theorem. Franklin [13, p. 268] also asserted that the generating functions are easily obtained.

Theorem 1.3 (*Franklin's Theorem*). *For any positive integer k and nonnegative integer m, the number of partitions of n with m distinct parts each occurring k or more times equals the number of partitions of n with exactly m distinct parts divisible by k.*

Thus by taking $m = 0$, Franklin's theorem degenerates to Glaisher's theorem, then by taking $k = 2$, Glaisher's theorem gives Euler's partition theorem.

From the works of Andrews and Chern, we noticed three conjectures posed by Beck concerning some variations of odd partitions and distinct partitions, which were only analytically proved by Andrews [4] and Chern [5] via differentiation technique in q-series introduced by Andrews [4]. In this paper, by extending Glaisher's bijection, we give the combinatorial proofs of these three conjectures directly or by proving more generalized results. For consistency of notations, we utilize the same notations in [4] and [5] in the rest of paper as far as possible.

Let $a(n)$ denote the number of partitions of n with only one even part which is possibly repeated. Beck [15] proposed the following conjecture:

Conjecture 1.1. *$a(n)$ is also the difference between the number of parts in the odd partitions of n and the number of parts in the distinct partitions of n.*

Let $c(n)$ denote the number of partitions of n in which exactly one part is repeated. Let $b(n)$ be the difference between the number of parts in the odd partitions of n and the number of parts in the distinct partitions of n. Andrews [4] analytically proved the following theorem by differentiation technique in q-series, which confirms the conjecture posed by Beck [15]:

Theorem 1.4 (*[4, Theorem 1]*). *For all $n \geq 1$, $a(n) = b(n) = c(n)$.*

Later Fu and Tang [9, Theorem 1.5] extended the result of Andrews and gave the analytic proof.

For $k \geq 1$, let $\mathcal{O}_k(n)$ be the set of partitions of n with no part divisible by k and $\mathcal{D}_k(n)$ be the set of partitions of n with no part occurring k or more times, respectively. Let $\mathcal{O}_{1,k}(n)$ be the partitions of n with exactly one part (possibly repeated) divisible by k. As one of main results in this paper, the following theorem generalizes Conjecture 1.1.

Theorem 1.5. *For $k \geq 2$ and $n \geq 0$, we have*

$$\left|\mathcal{O}_{1,k}(n)\right| = \frac{1}{k-1} \cdot \left(\sum_{\lambda \in \mathcal{O}_k(n)} \ell(\lambda) - \sum_{\lambda \in \mathcal{D}_k(n)} \ell(\lambda) \right).$$

Thus, letting $k = 2$ in Theorem 1.5 reduces the set $\mathcal{O}_{1,k}(n)$ to the set of partitions of n with only one even part which is possibly repeated, and the set $\mathcal{O}_k(n)$ (resp. $\mathcal{D}_k(n)$) to the set of odd (resp. distinct) partitions of n, which gives the combinatorial proof of Conjecture 1.1.

Let $a_1(n)$ denote the number of partitions of n such that there is exactly one part occurring three times while all other parts occur only once. Beck [15] made the following conjecture:

Conjecture 1.2. *$a_1(n)$ is also the difference between the number of parts in the distinct partitions of n and the number of distinct parts in the odd partitions of n.*

Let $b_1(n)$ be the difference between the total number of parts in the partitions of n into distinct parts and the total number of distinct parts in the partitions of n into odd parts. This conjecture was also proved by Andrews [4] with analytic method.