Applied Mathematics and Computation 342 (2019) 18–25

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Some fixed point theorems using weaker Meir–Keeler function in metric spaces with w−distance

Hossein Lakzian[a], B.E. Rhoades[b,*]

[a] Department of Mathematics, Payame Noor University, Tehran 19395-4697, Islamic Republic of Iran
[b] Department of Mathematics, Indiana University, Bloomington, IN 47405-7106, United States

ARTICLE INFO

MSC:
47H10
47H09

Keywords:
Fixed point
w-distance
Weakly contractive map
Meir–Keeler

ABSTRACT

In the present paper we prove some new fixed point theorems for self-mappings defined on a complete metric space with a w-distance. These results extend some previous fixed point theorems in this field to more general contractive conditions in the setting of w-distances for selfmappings which satisfy certain weaker Meir–Keeler conditions.

1. Introduction

In 2002 a fixed point theorem was established for mappings satisfying a contractive inequality of integral type by Branciari [3]. Rhoades [19] proved two fixed-point theorems for mappings satisfying a general contractive inequality of integral type. These results substantially extended the theorem of Branciari [3]. Jungck and Rhoades proved some fixed point theorems for weakly compatible maps on dislocated metric spaces [11]. Very recently, some new fixed point theorems for (φ, ψ, p)-weakly contractive mappings were established by Lakzian et al. [15,16] and Lakzian and Lin [14]. They generalized the results due to Banach [2], Branciari [3], Rhoades [20] and some other authors (see [6–8,10,18,21]). Some additional related papers are [1,4,12,13,23,24].

The concept of a w-distance on a metric space was introduced by Kada et al. [9]. They generalized Caristi's fixed point theorem, Ekeland's ϵ−variational's principle and the nonconvex minimization theorem of Takahashi.

2. Preliminaries

The purpose of this paper is to establish some fixed point theorems defined on a complete metric space with w-distance, and using contractive conditions of Meir–Keeler type. These results are generalizations of a number of results in the literature.

The following definition is the concept of w-distance on a metric space. (See Kada et al. [9]).

Definition 1 [9]**.** Let X be a metric space endowed with a metric d. A function $p : X \times X \longrightarrow [0, \infty)$ is called a w-distance on X if it satisfies the following properties:

(1) $p(x, z) \leq p(x, y) + p(y, z)$ for any $x, y, z \in X$;

* Corresponding author.
E-mail addresses: h_lakzian@pnu.ac.ir (H. Lakzian), rhoades@indiana.edu (B.E. Rhoades).

https://doi.org/10.1016/j.amc.2018.06.048
0096-3003/

(2) p is lower semi-continuous in its second variable; i.e., if $x \in X$ and $y_n \to y$ in X, then $p(x, y) \leq \liminf_n p(x, y_n)$;
(3) For each $\epsilon > 0$, there exists a $\delta > 0$ such that $p(z, x) \leq \delta$ and $p(z, y) \leq \delta$ imply $d(x, y) \leq \epsilon$.

We first recall the notion of a Meir–Keeler function (see [17]).

Definition 2 [17]**.** A function $\psi : [0, +\infty) \to [0, +\infty)$ is said to be a Meir–Keeler function if, for each $\eta > 0$, there exists a $\delta > 0$ such that, for $t \in [0, +\infty)$ with $\eta \leq t < \eta + \delta$, we have $\psi(t) < \eta$.

The definition of a weaker Meir–Keeler function is as follows:

Definition 3 [5]**.** We call $\psi : [0, +\infty) \to [0, +\infty)$ a weaker Meir–Keeler function if, for each $\eta > 0$, there exists a $\delta > 0$ such that, for $t \in [0, +\infty)$ with $\eta \leq t < \eta + \delta$, there exists an $n_0 \in \mathbb{N}$ such that $\psi^{n_0}(t) < \eta$.

As in [5], we shall assume a weaker Meir–Keeler function $\Psi := \{\psi : [0, +\infty) \to [0, +\infty)\}$ satisfying the following conditions:

(ψ_1) $\psi(t) > 0$ for $t > 0$ and $\psi(0) = 0$;
(ψ_2) for each $t \in [0, \infty)$, $\{\psi^n(t)\}_{n \in N}$ is decreasing;
(ψ_3) for $t_n \in [0, \infty)$, we have that
 (a) if $\lim_{n\to\infty} t_n = \gamma > 0$, then $\lim_{n\to\infty} \psi(t_n) < \gamma$, and
 (b) if $\lim_{n\to\infty} t_n = 0$, then $\lim_{n\to\infty} \psi(t_n) = 0$.

Also let $\Phi := \{\varphi : [0, +\infty) \to [0, +\infty)\}$ be the set of non-decreasing and continuous functions satisfying:

(φ_1) $\varphi(t) > 0$ for $t > 0$ and $\varphi(0) = 0$;
(φ_2) for each $\{t_n\} \subseteq (0, \infty)$, $\lim_{n\to\infty} t_n = 0$ if and only if $\lim_{n\to\infty} \varphi(t_n) = 0$.

Let p be a w-distance on a metric space (X, d), $\varphi \in \Phi$ and $\psi \in \Psi$. In what follows, we shall use φp (resp. ψp) to denote the composition of φ (resp. ψ) with p.

The following lemmas and theorems will be used in the next section.

Lemma 1 *[18]***.** *If $\psi \in \Psi$, then $\lim_n \psi^n(t) = 0$ for each $t > 0$ and, if $\varphi \in \Phi$, $\{a_n\} \subseteq [0, \infty)$ and $\lim_n \varphi(a_n) = 0$, then $\lim_n a_n = 0$.*

Lemma 2 *[9]***.** *Let (X, d) be a metric space and p be a w-distance on X.*

(i) *If $\{x_n\}$ is a sequence in X such that $\lim_n p(x_n, x) = \lim_n p(x_n, y) = 0$ then $x = y$. In particular, if $p(z, x) = p(z, y) = 0$ then $x = y$.*
(ii) *If $p(x_n, y_n) \leq \alpha_n$ $p(x_n, y) \leq \beta_n$ for any $n \in \mathbb{N}$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, \infty)$ converging to 0, then $\{y_n\}$ converges to y.*
(iii) *Let p be a w-distance on a metric space (X, d) and $\{x_n\}$ be a sequence in X such that, for each $\varepsilon > 0$, there exists an $N_\varepsilon \in N$ such that $m > n > N_\varepsilon$ implies $p(x_n, x_m) < \varepsilon$ (or $\lim_{m,n} p(x_n, x_m) = 0$) then $\{x_n\}$ is a Cauchy sequence.*

If $p(a, b) = p(b, a) = 0$ and $p(a, a) \leq p(a, b) + p(b, a) = 0$ then $p(a, a) = 0$ and by Lemma 2, $a = b$.

3. Main results

Define

$$M(x, y) = \max\left\{p(x, y), p(x, Tx), p(y, Ty), \frac{p(x, Ty) + p(Tx, y)}{2}\right\}; \tag{3.1}$$

and

$$m(x, y) = \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(Tx, y)}{2}\right\}. \tag{3.2}$$

The next theorem is one of the main results of this paper.

Theorem 3.1. *Let p be a w-distance on a complete metric space (X, d) such that $p(x, x) = 0$ for all $x \in X$. Suppose that T: $X \to X$ is such that, for all $x, y \in X$*

$$p(Tx, Ty) \leq M(x, y) - \varphi(M(x, y)), \tag{3.3}$$

where $\varphi \in \Phi$.

Suppose that either

(i) $\inf\{p(x, w) + p(x, Tx) : x \in X\} > 0$ *for every $w \in X$ with $w \neq Tw$,*
 or
(ii) *the mapping T is continuous.*