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Log-concavity of independence polynomials of some kinds of trees☆

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ABSTRACT

An *independent set* in a graph G is a set of pairwise non-adjacent vertices. Let $i_k(G)$ denote the number of independent sets of cardinality k in G. Then, its generating function

$$I(G;x)=\sum_{k=0}^{\alpha(G)} i_k(G)x^k$$

is called the *independence polynomial* of G (Gutman and Harary, 1983).

Alavi et al. (1987) conjectured that the independence polynomial of any tree or forest is unimodal. This conjecture is still open. In this paper, after obtaining recurrence relations and giving factorizations of independence polynomials for certain classes of trees, we prove the log-concavity of their independence polynomials. Thus, our results confirm the conjecture of Alavi et al. for some special cases.

1. Introduction

A graph polynomial is an algebraic object associated with a graph that is invariant under graph isomorphism. As such, it encodes information about the graph, and enables algebraic methods for extracting this information. In addition, it is also important in other subjects, i.e., both the matching polynomial and independence polynomial of a graph have many applications in chemistry and physics.

Let $G=(V,E)$ be a finite and simple graph. An *independent set* in G is a set of pairwise non-adjacent vertices. A *maximum independent set* in G is a largest independent set and its size is denoted by $\alpha(G)$ and called the independence number. Let $i_k(G)$ denote the number of independent sets of cardinality k in G. Then, its generating function

$$I(G;x)=\sum_{k=0}^{\alpha(G)} i_k(G)x^k,\quad i_0(G)=1$$

is called the *independence polynomial* of G (Gutman and Harary [9]).

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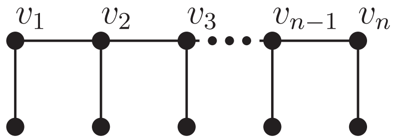

Fig. 1. W_n.

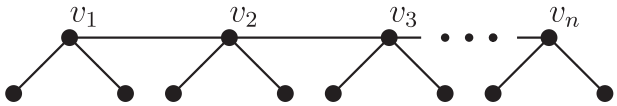

Fig. 2. H_n.

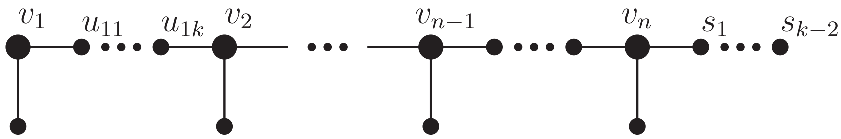

Fig. 3. A_n.

A polynomial $\sum_{k=0}^{n} a_k x^k$ with positive coefficients is called *unimodal* if there is some m, such that

$$a_0 \leq a_1 \leq \cdots \leq a_{m-1} \leq a_m \geq a_{m+1} \geq \cdots \geq a_n$$

and is called *log-concave* if $a_k^2 \geq a_{k-1}a_{k+1}$ for all $1 \leq k \leq n-1$ see [4,25]. It is known that log-concavity is a stronger property than unimodality. A basic approach to unimodality problems is to use Newton's inequalities: Let $a_0, a_1, \ldots, a_n$ be a sequence of nonnegative numbers. Suppose that the polynomial $\sum_{k=0}^{n} a_k x^k$ has only real zeros. Then

$$a_k^2 \geq a_{k-1}a_{k+1}\left(1+\frac{1}{k}\right)\left(1+\frac{1}{n-k}\right), \quad k = 1, 2, \ldots, n-1,$$

and the sequence is therefore log-concave and unimodal (see Hardy et al. [10]).

There are many unimodality problems on graph polynomials. In addition, unimodality problems in graph theory attract many researchers' great interest. For example, Huh [14] recently solved the long-standing open problems on the unimodality (Read [22]) and log-concavity (Welsh [27]) of the chromatical polynomial of a graph. Heilmann and Lieb [13] proved that the matching polynomial of a graph is unimodal since it has only real zeros. Unimodality problems of independence polynomials have been extensively studied, see [1,2,5,6,8,12,17–19,21,26,29,30] for instance. In general, the independence polynomial of a graph may be neither log-concave nor unimodal, as evidenced by the graph $G = 3K_4 + K_{37}$ with $I(G; x) = 1 + 49x + 48x^2 + 64x^3$. But the independence polynomials for certain special classes of graphs are unimodal and even have only real zeros. For instance, the independence polynomial of a line graph has only real zeros [13]. More generally, the independence polynomial of a claw-free graph has only real zeros [8]. In addition, Alavi, Malde, Schwenk, Erdős [1] conjectured the following.

Conjecture 1.1. *The independence polynomial of any tree or forest is unimodal.*

This conjecture is still open and only has minor progress. For example, for any path P_n, any centipede graph W_n (Fig. 1), any caterpillar graph H_n (Fig. 2), any vertebrated graph and any firecracker graph, respectively, its independence polynomial is unimodal, see Levit and Mandrescu [15,16], Zhu [28], Wang and Zhu [26]. For any tree T, the independence polynomial of the corona of T and $2K_1$ is unimodal, see Zhu [29] for instance. We refer the reader to [23,24,29] for more results from operations of graphs.

Motivated by Conjecture 1.1, we study the unimodality property of independence polynomials of some trees. First, let us introduce the trees studied in this paper.

Definition 1.1. Let G be a graph with a root v . A new graph, denoted by $G_{n,\,k,\,m}$, is obtained from G and the path P_{n+1} as follows:

(i) insert k vertices $u_{i1}, u_{i2}, \ldots, u_{ik}$ into each edge $v_i v_{i+1}$ of P_{n+1} for $1 \leq i \leq n-1$ and insert $m-1$ vertices into the last edge of P_{n+1} ;
(ii) identify the vertex v_i of P_{n+1} with the root v of ith copy of G for $1 \leq i \leq n$, respectively.

For example, take $G = K_2$ with its one vertex as the root and denote the graph $G_{n,k,k-2}$ by A_n (see Fig. 3), and take $k = 2$ and $G = K_{1,t}$ with its center as the root and denote the graph $G_{n,\,2,\,0}$ by $D_{n,\,t}$ (see Fig. 4 for $D_{n,\,t}$), where t is a nonnegative integer.