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A high-order $L2$-compact difference method for Caputo-type time-fractional sub-diffusion equations with variable coefficients☆

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ABSTRACT

A high-order compact finite difference method is proposed for solving a class of time-fractional sub-diffusion equations. The diffusion coefficient of the equation may be spatially variable and the time-fractional derivative is in the Caputo sense with the order $\alpha \in (0, 1)$. The Caputo time-fractional derivative is discretized by a $(3-\alpha)$ th-order numerical formula (called the $L2$ formula here) which is constructed by piecewise quadratic interpolating polynomials but does not require any sub-stepping scheme for the approximation at the first-time level. The variable coefficient spatial differential operator is approximated by a fourth-order compact finite difference operator. By developing a technique of discrete energy analysis, a full theoretical analysis of the stability and convergence of the method is carried out for the general case of variable coefficient and for all $\alpha \in (0, 1)$. The optimal error estimate is obtained in the L^2 norm and shows that the proposed method has the temporal $(3-\alpha)$ th-order accuracy and the spatial fourth-order accuracy. Further approximations are also considered for enlarging the applicability of the method while preserving its high-order accuracy. Applications are given to three model problems, and numerical results are presented to demonstrate the theoretical analysis results.

1. Introduction

As a class of basic fractional differential equations, time-fractional sub-diffusion equations (TFSDEs for short) have significant applications in a broad range of fields; see, e.g., [1] and the references therein. This kind of equation was derived by using continuous time random walks with a fractional derivative term in time to represent the degree of memory in the diffusing material [2]. In the inhomogeneous medium, the diffusion coefficient of the equation may be spatially variable. This leads to numerous applications which are described by TFSDEs involving variable diffusion coefficients (see, e.g., [3–7]). In this paper, we seek a high-order numerical method for computing numerical solutions of a class of Caputo-type TFSDEs

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with variable coefficients. Consider the following initial-boundary value problem

$$\begin{cases} {}_0^C\mathcal{D}_t^\alpha u(x,t) = \mathcal{L}u(x,t) + f(x,t), & (x,t) \in D \equiv (0,L) \times (0,T], \\ u(0,t) = \phi_0(t), \quad u(L,t) = \phi_L(t), & t \in (0,T], \\ u(x,0) = \psi(x), \quad x \in [0,L], \end{cases} \tag{1.1}$$

where the term ${}_0^C\mathcal{D}_t^\alpha u(x,t)$ represents the Caputo fractional derivative of order α in t, which is defined by

$${}_0^C\mathcal{D}_t^\alpha u(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial_s u(x,s)}{(t-s)^\alpha} \mathrm{d}s, \qquad 0 < \alpha < 1, \tag{1.2}$$

and the term $\mathcal{L}u(x,t)$ is the diffusion term with the diffusion coefficient $k(x)$, which is given by

$$\mathcal{L}u(x,t) = \partial_x (k(x)\partial_x u)(x,t). \tag{1.3}$$

Throughout the paper, we shall assume that the given functions f, ϕ_0, ϕ_L, ψ and k in (1.1) and (1.3) are smooth enough and there exist positive constants κ_0 and κ_1 such that

$$\kappa_0 \le k(x) \le \kappa_1, \qquad x \in [0,L]. \tag{1.4}$$

In addition, we assume that the solution to the problem (1.1) has the necessary regularity.

The existing numerical methods for solving TFSDEs were mostly derived for the equations with constant diffusion coefficients. For these methods, we refer to [8–14] and related references therein. On the other hand, there are also a few works that are devoted to numerical methods for variable coefficient problems of TFSDEs in the form (1.1). Zhao and Xu [4] proposed a compact finite difference method and proved the unconditional stability and convergence of the method by introducing a new norm regarding to the variable coefficient $k(x)$. In the recent work [7], Vong et al. extended the method in [4] to the Neumann problem of (1.1). By applying the energy method to the matrix form of the method, the proposed method was shown to be stable and convergent for the general case of variable coefficient $k(x)$. Cui [5] considered a compact exponential finite difference method for a time-fractional convection–diffusion reaction problem with variable coefficients that contains the present problem (1.1) as a special case. However, the related convergence analysis given there is only for the special case of constant coefficients. Similarly in [6], Cui proposed a combined compact finite difference method in a general setting, and some theoretical results for the equation of integer order with constant coefficients were obtained. In the above works, the traditional $L1$ formula was used for the approximation of the Caputo fractional derivative ${}_0^C\mathcal{D}_t^\alpha u$ so that the temporal accuracy of the method is only of order $2-\alpha$, which is less than two. For the time-space variable diffusion coefficient problems of TFSDEs, Mustapha et al. [15] investigated a piecewise linear time-stepping discontinuous Galerkin method combined with the standard Galerkin finite element scheme in space. The well-posedness of the fully discrete scheme and error analysis were shown. Recently in [16], Mustapha studied a semidiscrete Galerkin finite element method and the main focus is on achieving optimal error results. In this paper, we shall present a high-order compact finite difference method, which possesses the temporal accuracy of order $3-\alpha$ (higher than second-order) and the spatial fourth-order accuracy, for the variable coefficient problem (1.1). Also we shall establish a full analysis of stability and convergence of the proposed method for the general variable coefficient $k(x)$ and for all $\alpha \in (0, 1)$.

The main idea of the traditional $L1$ formula for approximating Caputo fractional derivative ${}_0^C\mathcal{D}_t^\alpha y$ of the function $y(t)$ is to replace the integrand $y(t)$ inside the integral by its piecewise linear interpolating polynomial (see [17]). A simple technique for improving the accuracy of $L1$ formula is to use piecewise high-degree interpolating polynomials instead of the linear interpolating polynomial. In general, the obtained numerical formulae in this way improve the accuracy of $L1$ formula form the order $2-\alpha$ to the order $r+1-\alpha$, where $r \ge 2$ is the degree of the interpolating polynomial. When such formulae are applied to solve TFSDEs, a key issue is the stability analysis of the corresponding methods for all $\alpha \in (0, 1)$. Using piecewise quadratic interpolating polynomials, Alikhanov [18] derived a numerical formula (called $L2-1_\sigma$ formula) to approximate the Caputo fractional derivative ${}_0^C\mathcal{D}_t^\alpha y$ at a special point with the numerical accuracy of order $3-\alpha$. Then some finite difference methods based on the $L2-1_\sigma$ formula were proposed for solving TFSDEs; but the methods are only second-order accurate in time. Gao et al. [19] constructed a so-called $L1-2$ formula for the Caputo fractional derivative ${}_0^C\mathcal{D}_t^\alpha y$ by means of the linear interpolating polynomial on the first subinterval and the quadratic interpolating polynomials on the other subintervals. The applications of the $L1-2$ formula to solving TFSDEs were exhibited in that paper through a variety of test examples. Li's group [20,21] extended the $L1-2$ formula by using more higher-degree interpolating polynomials and derived a series of high-order formulae for the approximation of the Caputo fractional derivative ${}_0^C\mathcal{D}_t^\alpha y$. In those works, the applications of the derived formulae to a time-fractional advection–diffusion equation with constant coefficients were discussed. A analysis of the stability and convergence of the resulting methods was also carried out by the Fourier method for α within a certain region in (0,1). It remains unknown how to prove theoretically the stability and convergence of the methods for all $\alpha \in (0, 1)$. Recently, Lv and Xu [12] constructed a numerical formula for the Caputo fractional derivative ${}_0^C\mathcal{D}_t^\alpha y$ by using piecewise quadratic interpolating polynomials. This formula is slightly different from the $L2-1_\sigma$ and $L1-2$ formulae; but it has still the numerical accuracy of order $3-\alpha$. More interestingly, a novel technique was developed in [12] so that a full analysis of stability and convergence of the $(3-\alpha)$ th-order method for TFSDEs with constant coefficients was carried out for all $\alpha \in (0, 1)$.

In this paper, we apply the numerical formula in [12], called $L2$ formula here, to the time approximation of the present variable coefficient problem (1.1). In order to obtain a global $(3-\alpha)$ th-order accurate scheme, it was suggested in [12] that