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Stability and convergence of characteristic MAC scheme and post-processing for the Oseen equations on non-uniform grids

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ABSTRACT

In this paper, we establish the LBB condition and stability for both velocity and pressure of characteristic MAC scheme for the Oseen equations on non-uniform grids. We obtain the second order convergence in discrete L^2 norm for both velocity and pressure and the first order convergence in discrete H^1 norm for velocity. Moreover, we construct the post-processing characteristic MAC scheme to obtain second order accuracy in discrete H^1 norm for the velocity. Finally, some numerical experiments are presented to show the correctness and accuracy of the MAC scheme.

1. Introduction

The Stokes and Navier-Stokes equations are important basic models in flow dynamics and often employed in scientific computation. They have been widely applied in physics, chemistry and engineering. Usually, it is difficult to get analytical solutions of the Stokes and Navier-Stokes equations. Thus, numerical methods become a powerful tool for solving these equations.

Among the many numerical methods that are available for solving the Stokes and Navier- Stokes equations, one of the best and simplest is the marker and cell method (Welch et al. [1]). The same point view has been shared by Han and Wu [2]: “It is well known that the marker and cell (MAC) is one of the simplest and most effective numerical schemes for solving Stokes equations and Navier-Stokes equations.” The MAC method has been widely used in engineering applications as evidenced being the basis of many flow packages [3]. The MAC scheme has the ability to enforce the incompressibility constraint of the velocity field point-wisely. Moreover, it has been shown to locally conserve the mass, momentum and kinetic energy [4,5]. The MAC method is a class of finite volume method on rectangular cells with pressure approximated at the cell center, the x-component of velocity approximated at the midpoint of vertical edges of the cell, and the y-component of velocity approximated at the midpoint of horizontal edges of the cell. The MAC method can also be interpreted as a mixed finite-element method coupled with a quadrature formula. This interpretation can be found in Girault and Raviart [6,7], where the mixed method is analyzed. And the error estimates for the MAC method were derived in [8].

The numerical solution of the non-stationary, incompressible Navier-Stokes model can be split into linearized auxiliary problems of Oseen type. Hamilton and his coauthors [9] investigated the performance of smoothers based on the

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Hermitian/skew-Hermitian and augmented Lagrangian splittings applied to the MAC discretization of the Oseen problem. In 2015, error estimates with the optimal convergence order are proved by Notsu and Tabata [10] for a pressure stabilized characteristics finite element scheme for the Oseen equations. Stabilized finite element methods for the generalized Oseen problem have been studied by Braack and his coauthors [11]. Some extensions to the algebraic multigrid method based on element interpolation and the application to the mixed finite element discretization of the Oseen linearized Navier-Stokes equations have been presented in [12]. In 2007, several preconditioners for the pressure Schur complement of the discrete steady Oseen problem were considered in [13]. Heister and Rapin have presented a novel approach leveraging stabilization for inf-sup stable discretizations for the Oseen equations in [14]. Conforming finite element approximations of the time-dependent Oseen problem with inf-sup stable approximation of velocity and pressure were studied in [15]. In 2016, Barrios and his coauthors [16] proposed a new augmented dual-mixed method for the Oseen problem based on the pseudostress-velocity formulation.

In many diffusion processes arising in physical problems, convection essentially dominates diffusion. The mathematical modeling of convection-diffusion equations arises in many technological and scientific engineering applications. For convection-dominated diffusion problem, standard finite difference methods or finite element methods perform poorly and may exhibit nonphysical oscillations. In order to deal with convection-dominated problems, a lot of ideas have been proposed, e.g., upwind methods [11,17–19], characteristic methods [20–24] and so on. Our scheme belongs to the second group. Compared with the traditional finite difference methods or finite element methods, methods of characteristics have much smaller time-truncation errors to obtain the same accuracy in problems with significant convection. These schemes of characteristics have the significant practicability with permitting the use of large time steps and achieving improvements in efficiency. Furthermore, it can overcome nonphysical oscillations.

In this paper, we establish the LBB condition and stability for both velocity and pressure of characteristic MAC scheme for the Oseen equations on non-uniform grids. Inspired by the analysis technique in [24–32], we obtain the second order convergence in discrete L^2 norm for both velocity and pressure and the first order convergence in discrete H^1 norm for velocity. Our approach for error estimates here is different from that in [30]. More specifically, we only need to test the scheme with the error of velocity function to obtain the second order accuracy in space due to a simpler estimate approach to the characteristic curves. Besides, we obtain the second order accuracy in space without assuming that $\Delta t \leq C(h^2+k^2)$, where Δt is the time step, h and k are maximal mesh sizes of x and y-directional grids. Stability results are proven rigorously and carefully which are not given in [30]. Moreover, we construct the post-processing characteristic MAC scheme to obtain second order accuracy in discrete H^1 norm for the velocity. Finally, some numerical experiments are presented to show the correctness and accuracy of the MAC scheme. We also give some comparison of computational times and accuracy between the characteristic MAC method and the standard MAC scheme to show the advantages of the use of characteristic curves.

The paper is organized as follows. In Section 2 we give the problem and some preliminaries. In Section 3 we present the characteristic MAC scheme and stability. In Section 4 we demonstrate error analysis for discrete scheme. In Section 5 we construct the post-processing MAC scheme to obtain second order accuracy in discrete H^1 norm for the velocity. In Section 6 some numerical experiments using the MAC schemes are carried out, which show the accuracy and advantages of the characteristic MAC scheme.

Throughout the paper we use C, with or without subscript, to denote a positive constant, which could have different values at different appearances.

2. The problem and some preliminaries

In this paper, we consider the time dependent Oseen equations with homogenerous boundary and initial condition in two dimensional domain for an incompressible fluid.

Find p and $\mathbf{u}=(u^x,u^y)$ such that [10]

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t}+\boldsymbol{\omega}\cdot\nabla\mathbf{u}-\mu\Delta\mathbf{u}+\nabla p+\mathbf{A}\mathbf{u}=\mathbf{f}, & in\ \Omega\times J,\\ \nabla\cdot\mathbf{u}=0, & in\ \Omega\times J,\\ \mathbf{u}=0, & on\ \partial\Omega\times J,\\ \mathbf{u}(x,y,0)=\mathbf{u}_0(x,y), & in\ \Omega\times J. \end{cases} \tag{1}$$

Here, $\mathbf{u}=(u^x,u^y)$ and p denote the unknown velocity vector and pressure fields. $\mu=\frac{1}{Re}>0$, where Re is the Reynolds number. $\boldsymbol{\omega}$ is a divergence free advection velocity. $\mathbf{A}(x,y,t)=\begin{pmatrix} a_1(x,y,t) & \\ & a_2(x,y,t)\end{pmatrix}$ is a given reaction rate. $\mathbf{f}=(f^x,f^y)\in(L^2(\Omega))^2$, represents the source term. For simplicity, we take $\Omega=(0,L_x)\times(0,L_y)$ as a rectangular domain. $J=(0,T]$, and T denotes the final time.

Firstly, we give the partitions and notation as follows.

Let $N>0$ be a positive integer. Set

$$\Delta t=T/N,\quad t^n=n\Delta t,\ \ for\ n\leq N.$$

The two dimensional domain Ω is partitioned by $\delta_x\times\delta_y$, where

$$\delta_x: 0=x_0<x_1<\cdots<x_{N_x-1}<x_{N_x}=L_x,$$