Applied Mathematics and Computation 342 (2019) 112–117

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On the antimaximum principle for the discrete p-Laplacian with sign-changing weight

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ARTICLE INFO

Keywords:
Difference equations
Discrete p-Laplacian
Maximum principle
Antimaximum principle
Eigenvalue
Eigenfunction

ABSTRACT

This work deals with the antimaximum principle for the discrete Neumann and Dirichlet problem

$$-\Delta\varphi_p(\Delta u(k-1)) = \lambda m(k)|u(k)|^{p-2}u(k) + h(k) \quad \text{in} \quad [1, n].$$

We prove the existence of three real numbers $0 \leq a < b < c$ such that, if $\lambda \in]a, b[$, every solution u of this problem is strictly positive (maximum principle), if $\lambda \in]b, c[$, every solution u of this problem is strictly negative (antimaximum principle) and if $\lambda = b$, the problem has no solution. Moreover these three real numbers are optimal.

1. Introduction

This paper is concerned with the Neumann or Dirichlet problem

$$-\Delta\varphi_p(\Delta u(k-1)) = \lambda m(k)|u(k)|^{p-2}u(k) + h(k) \quad \text{in} \quad [1, n],$$

where n is an integer greater than or equal to 1, $[1, n]$ is the discrete interval $\{1, \dots, n\}$, $\Delta u(k) := u(k+1) - u(k)$ is the forward difference operator, $\varphi_p(s) = |s|^{p-2}s$, $1 < p < \infty$, h function defined on $[1, n]$ and m changes sign in $[1, n]$. The original form for the antimaximum principle concerns the continuous problem

$$-\Delta_p u = \lambda m(x)|u|^{p-2}u + h(x) \text{ in } \Omega, Bu = 0 \text{ on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^N$, $\Delta_p u := div(|\nabla u|^{p-2}\nabla u)$, is the p-Laplacian and $Bu = 0$ represents either the Dirichlet or the Neumann homogeneous boundary conditions (see [5]). The argument here uses a discrete forme of Picone's identity (see [3]). Some of our arguments are inspired by [2,6]. In an article submitted [7], we studied the existence and nonexistence of positive solution and its uniqueness depending on the sign of $\sum_{k=1}^{n} m(k)$ and on whether or not λ belongs to $]0, \mu(m)[$ in the Neumann case, and depending whether or not λ belongs to $]\lambda_{-1}(m), \lambda_1(m)[$ in the Dirichlet case, where $\mu(m)$, $\lambda_1(m)$ and $\lambda_{-1}(m)$ are defined in (2.6) and (2.10). To give an idea of our results, let us consider the Neumann problem (2.1), with $\sum_{k=1}^{n} m(k) \leq 0$, we show that the antimaximum principle (in brief AMP) holds at the right of $\mu(m)$ and the left of 0. Moreover, it is uniform and the intervals of uniformity are exactly $\mu(m) < \lambda < \mu^*(m)$ and $-\mu^*(-m) < \lambda < 0$, where $\mu^*(m)$ is defined in (3.1). We will also observe that the AMP cannot hold far away to the right of $\mu^*(m)$ or to the left of $-\mu^*(-m)$.

We do the same for the Dirichlet problem (2.8), with $\sum_{k=1}^{n} m(k) \leq 0$, we show that the AMP holds at the right of $\lambda_1(m)$ and the left of $\lambda_{-1}(m)$. Moreover, it is uniform and the intervals of uniformity are exactly $\lambda_1(m) < \lambda < \lambda^*(m)$ and

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https://doi.org/10.1016/j.amc.2018.09.012
0096-3003/

$-\lambda^*(-m) < \lambda < \lambda_{-1}(m)$, where $\lambda^*(m)$ is defined in (4.1). We will also observe that the AMP cannot hold far away to the right of $\lambda^*(m)$ or to the left of $-\lambda^*(-m)$.

2. Preliminaries

Consider the Neumann problem

$$\begin{cases} -\Delta\varphi_p(\Delta u(k-1)) = \lambda m(k)|u(k)|^{p-2}u(k) + h(k) \quad \text{in} \quad [1,n], \\ \Delta u(0) = \Delta u(n) = 0. \end{cases} \tag{2.1}$$

Suppose that

$$\exists k_1,\ k_2 \in [1,n];\ m(k_1)m(k_2) < 0. \tag{2.2}$$

Also, without loss of generality, we can assume that

$$|m(k)| < 1, \quad \forall k \in [1,n]. \tag{2.3}$$

The class $W = \{u : [0, n+1] \to \mathbb{R}\ ;\ \Delta u(0) = \Delta u(n) = 0\}$ is an n-dimensional space under the norm $\|u\| = (\sum_{k=1}^{n} |u(k)|^p)^{1/p}$.

Consider the nonlinear eigenvalue Neumann problem

$$\begin{cases} -\Delta\varphi_p(\Delta u(k-1)) = \lambda m(k)|u(k)|^{p-2}u(k)) \quad \text{in} \quad [1,n], \\ \Delta u(0) = \Delta u(n) = 0. \end{cases} \tag{2.4}$$

Proposition 2.1. *Let u be a solution of*

$$\begin{cases} -\Delta\varphi_p(\Delta u(k-1)) + a_0(k)|u(k)|^{p-2}u(k) = h(k) \quad \text{in} \quad [1,n], \\ \Delta u(0) = \Delta u(n) = 0, \end{cases} \tag{2.5}$$

where $a_0 \geq 0$ *and* $h \gneqq 0$. *Then* $u > 0$ *in* $[1, n]$.

Proof. Writing $u = u^+ - u^-$ with $u^\pm = \max\{\pm u, 0\}$ and taking $-u^-$ as testing function in (2.5),

$$-\sum_{k=1}^{n} \varphi_p(\Delta u(k-1))\Delta u^-(k-1) + \sum_{k=1}^{n} a_0(k)|u^-(k)|^p = -\sum_{k=1}^{n} h(k)u^-(k).$$

Distinguishing the cases of sign of $u(k-1)$ and $u(k)$, we prove that

$$\sum_{k=1}^{n} |\Delta u^-(k-1)|^p \leq -\sum_{k=1}^{n} \varphi_p(\Delta u(k-1))\Delta u^-(k-1),$$

then

$$\sum_{k=1}^{n} |\Delta u^-(k-1)|^p + \sum_{k=1}^{n} a_0(k)|u^-(k)|^p \leq -\sum_{k=1}^{n} h(k)u^-(k) \leq 0,$$

therefore $\sum_{k=1}^{n} |\Delta u^-(k-1)|^p = 0$ and u^- is constant. If $u^- \neq 0$, since $\sum_{k=1}^{n} h(k)u^-(k) = 0$, then $h \equiv 0$ which is absurd. Thus $u \geq 0$.

On the other hand, if $u(k_0) = 0$ for some $k_0 \in [1, n]$, then $\Delta u(k_0) = u(k_0+1) \geq 0$ and $\Delta u(k_0-1) = -u(k_0-1) \leq 0$, so $\varphi_p(\Delta u(k_0)) \geq 0$ and $\varphi_p(\Delta u(k_0-1)) \leq 0$. As $-\varphi_p(\Delta u(k_0)) + \varphi_p(\Delta u(k_0-1)) + a_0(k_0)(u(k_0))^{p-1} = h(k_0) \geq 0$, then $0 \leq \varphi_p(\Delta u(k_0)) \leq \varphi_p(\Delta u(k_0-1)) \leq 0$, from where $u(k_0+1) = u(k_0-1) = 0$ and so on, we prove $u \equiv 0$, which contradicts $h \neq 0$. □

Corollary 2.2. *If* $u \gneqq 0$ *is a solution of (2.1) with* $h \geq 0$, *then* $u > 0$.

Proof. By writing Eq. (2.1) as

$$-\Delta\varphi_p(\Delta u(k-1)) \pm \lambda|u(k)|^{p-2}u(k) = \lambda(m(k) \pm 1)|u(k)|^{p-2}u(k) + h(k),$$

and using (2.3); (here + is used if $\lambda \geq 0$, - if $\lambda < 0$), we conclude by Proposition 2.1. □

The following expression will play a central role in our approach:

$$\mu(m) := \inf\left\{ \sum_{k=1}^{n} |\Delta u(k-1)|^p :\ u \in W \ \text{ and } \ \sum_{k=1}^{n} m(k)|u(k)|^p = 1 \right\}. \tag{2.6}$$

Proposition 2.3. *(i) Suppose that* $\sum_{k=1}^{n} m(k) < 0$. *Then* $\mu(m) > 0$, *every eigenfunction with* $\mu(m)$ *of (2.4) does not change sign in* $[1, n]$ *and does not vanish in* $[1, n]$, *and* $\mu(m)$ *is the unique nonzero principal eigenvalue of (2.4); moreover, the interval* $]0, \mu(m)[$ *does not contain any eigenvalue of (2.4).*