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Unconditionally stable compact theta schemes for solving the linear and semi-linear fourth-order diffusion equations

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ABSTRACT

This paper is concerned with numerical methods for solving a class of fourth-order diffusion equations. Combining the compact difference operator in space discretization and the linear θ method in time, the compact theta scheme for the linear problem is first proposed. By virtue of the Fourier method, the suggested scheme is shown to be unconditionally stable and convergent in the discrete L^2-norm for any $\theta \in [1/2, 1]$. And then this idea is generalized to the semi-linear case, the corresponding compact theta scheme is constructed and analyzed in detail. Numerical experiments corresponding to the linear and semi-linear situations are carried out to support our theoretical statements.

1. Introduction

In the past few decades, much considerable work has been done theoretically or numerically on the second-order diffusion equations, i.e., the conventional parabolic partial differential equations. But in some applications, a fourth order space derivative term must be indispensable. For example, the wave propagation in beams and modeling formation of grooves on a flat surface because of grain require fourth-order space derivative terms in their formulations [1,2]. Moreover, it is used to optimize the trade-off between noise removal and edge preservation in image processing and nuclear medicine, see [3,4]. In addition, the fourth order space derivative term also appear in the Cahn–Hilliard equations which was originally introduced by Cahn and Hilliard in [5] as a model for isothermal phase separation phenomena in binary mixtures.

In this paper, firstly, we consider the following fourth-order initial-boundary value problems

$$\frac{\partial u}{\partial t} + b^2 \frac{\partial^4 u}{\partial x^4} = f(x,t), \quad 0 < x < L, \quad 0 < t \le T, \tag{1.1}$$

$$u(x,0) = \phi(x), \quad 0 \le x \le L, \tag{1.2}$$

$$u(0,t) = u_0(t), \quad u(L,t) = u_L(t), \quad 0 \le t \le T, \tag{1.3}$$

$$\frac{\partial^2 u}{\partial x^2}(0,t) = \eta_0(t), \quad \frac{\partial^2 u}{\partial x^2}(L,t) = \eta_L(t), \quad 0 \le t \le T, \tag{1.4}$$

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where u_0, u_L, η_0, η_L are given smooth functions. In [6], Kwembe discussed the existence and uniqueness of weak global smooth solutions of the parabolic equation of the bi-harmonic type in the Zheng-Li Banach space. In [7], King et al. studied the continuum model for epitaxial thin film growth and showed the existence, uniqueness and regularity of solutions in an appropriate function space. In [8], Xu and Zhou established the existence and uniqueness of weak solutions for the initial-boundary value problem of a fourth-order nonlinear parabolic equation.

Nevertheless, the exact solutions of such equations are not easy to obtain in general and relatively few works have been done on the numerical solution. And for all we know, most of the existing work deal with the fourth-order linear parabolic equations, see e.g., [9,10]. For example, the explicit and implicit finite difference schemes for the fourth-order parabolic equations have been proposed by Conte [11], Mohanty et al. [12] and Evans [13], respectively. The alternating group explicit iterative method, the spline method and some approximations with high accuracy have been applied to the parabolic equations, see [14–17]. However, the fourth-order diffusion equations considered here have not previously been studied, especially the nonlinear case. In this paper, we generalize the linear theta method to the linear and semi-linear fourth-order problems (1.1)–(1.4) and establish the unconditionally stable compact theta schemes. The linear theta method is originally applied to solve numerically ordinary differential equations (see, e.g., [18,19]), and it has recently been extended to the partial differential equation with piecewise continuous arguments [20], the stochastic differential equations [21] and the diffusion equations with time delay [22,23].

The main novelty of our work is that the suggested compact theta schemes are unconditionally stable and convergent for any $\theta \in [1/2, 1]$. That is, the stability and convergence of the suggested schemes are not affected by the step selection. Secondly, both compact theta schemes have fourth order accuracy in space and the computational overheads are competitive since the coefficient matrices of both compact schemes have the same band-width compared with the corresponding non-compact schemes with second order accuracy, see e.g., [24]. Furthermore, this work provides an effective way for solving numerically the semi-linear problem.

The rest of the paper is organized as follows. In Section 2, we propose the compact theta scheme for the linear problems (1.1)–(1.4). In Section 3, we first discuss the solvability, the local truncation error and the consistency of the suggested scheme. And then we prove the stability and convergence by using the Fourier method. In Section 3, we generalize this idea to the semi-linear case, and construct and analyze the corresponding compact theta scheme. In the last section, two numerical experiments are given to illustrate the effectiveness and accuracy of the suggested schemes.

2. Compact theta scheme

In the section we are devoted to deriving the compact theta scheme for the linear initial-boundary value problems (1.1)–(1.4).

Let $h = L/M, \tau = T/N$ be the uniform space and time mesh sizes for positive integers M and N. Denote $x_j = jh, t_k = k\tau$, $\Omega_h = \{x_j | 0 \le j \le M\}$ and $\Omega_\tau = \{t_k | 0 \le k \le N\}$. Suppose $\mathcal{V} = \{v | \ v = (v_1^k, v_2^k, \ldots, v_M^k), 0 \le k \le N\}$ is the grid function space defined on $\Omega_{h\tau} = \Omega_h \times \Omega_\tau$, and $\mathring{\mathcal{V}} = \{v | \ v \in \mathcal{V}, v_0^k = v_M^k = 0\}$. For any function $v \in \mathring{\mathcal{V}}$, we define

$$v_j^{k-1/2} = \frac{1}{2}(v_j^k + v_j^{k-1}), \quad \delta_t v_j^{k-1/2} = \frac{1}{\tau}(v_j^k - v_j^{k-1}), \quad \delta_x^2 v_j^k = \frac{1}{h^2}(v_{j-1}^k - 2v_j^k + v_{j+1}^k), \quad \delta_x^4 v_j^k = \delta_x^2(\delta_x^2 v_j^k), \tag{2.1}$$

and

$$\|v^k\| = \sqrt{h\sum_{j=1}^{M-1}(v_j^k)^2}. \tag{2.2}$$

Considering the problem (1.1) at the point $(x_j, t_{k-\frac{1}{2}})$ yields

$$\frac{\partial u}{\partial t}(x_j, t_{k-\frac{1}{2}}) + b^2\frac{\partial^4 u}{\partial x^4}(x_j, t_{k-\frac{1}{2}}) = f(x_j, t_{k-\frac{1}{2}}), \quad 1 \le j \le M, \quad 1 \le k \le N. \tag{2.3}$$

Denote

$$U_j^k = u(x_j, t_k), \quad z(x,t) = \frac{\partial^4 u}{\partial x^4}(x,t), \quad z_j^k = z(x_j, t_k). \tag{2.4}$$

Suppose $u(x,t) \in C_{x,t}^{8,3}$, using Taylor expansion and (2.4), we have

$$\begin{aligned}
\delta_x^4 U_j^k &= \frac{\partial^4 u}{\partial x^4}(x_j, t_k) + \frac{h^2}{6}\frac{\partial^6 u}{\partial x^6}(x_j, t_k) + \frac{h^4}{80}\frac{\partial^8 u}{\partial x^8}(\eta_j, t_k) = z_j^k + \frac{h^2}{6}\frac{\partial^2 z}{\partial x^2}(x_j, t_k) + \frac{h^4}{80}\frac{\partial^4 z}{\partial x^4}(\eta_j, t_k) \\
&= z_j^k + \frac{h^2}{6}\left[\delta_x^2 z_j^k - \frac{h^2}{12}\frac{\partial^4 z}{\partial x^4}(\zeta_j, t_k)\right] + \frac{h^4}{80}\frac{\partial^4 z}{\partial x^4}(\eta_j, t_k) = \left(1 + \frac{h^2}{6}\delta_x^2\right)z_j^k + \frac{h^4}{720}\frac{\partial^4 z}{\partial x^4}(\varsigma_j, t_k) \\
&= \frac{1}{6}(z_{j-1}^k + 4z_j^k + z_{j+1}^k) + \frac{h^4}{720}\frac{\partial^4 z}{\partial x^4}(\varsigma_j, t_k), \quad 1 \le j \le M-1,
\end{aligned} \tag{2.5}$$

where $\eta_j, \zeta_j, \varsigma_j \in (x_{j-2}, x_{j+2})$.