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# Radial basis functions method for solving the fractional diffusion equations

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ABSTRACT

Fractional order diffusion equations are generalizations of classical diffusion equations, treating super-diffusive flow processes. The paper presents a meshless method based on spatial trial spaces spanned by the radial basis functions (RBFs) for the numerical solution of a class of initial-boundary value fractional diffusion equations with variable coefficients on a finite domain. The space fractional derivatives are defined by using Riemann–Liouville fractional derivative. We first provide Riemann–Liouville fractional derivatives for the five kinds of RBFs, including the Powers, Gaussian, Multiquadric, Matérn and Thin-plate splines, in one dimension. The time-dependent fractional diffusion equation is discretized in space with the RBF collocation method and the remaining system of ordinary differential equations (ODEs) is advanced in time with an ODE method using a method of lines approach. Some numerical results are given in order to demonstrate the efficiency and accuracy of the method. Additionally, some physical properties of this fractional diffusion system are simulated, which further confirm the effectiveness of our method. The stability of the linear systems arising from discretizing Riemann–Liouville fractional differential operator with RBFs is also analysed.



## 1. Introduction

The basic idea behind fractional calculus has a history similar to that of more classical calculus and the topic has attracted the interests of mathematicians who contributed fundamentally to the development of classical calculus [6]. During the last decade, fractional calculus emerges increasingly as a tool for the description of a broad range of non-classical phenomena in the applied sciences and engineering. Fractional-order differential operators have been used to model a wide range of problems in surface and subsurface hydrology [1,2,22,58,59], plasma turbulence [19,20], finance [8,67], epidemiology [4,35] and ecology [12,33].

Diffusion processes in complex systems are often observed to deviate from standard laws. The discrepancies can occur both for the time relaxation that can deviate from the classical exponential pattern and for the spatial diffusion that can deviate from Ficks second law. The resulting diffusion processes are no longer Brownian and cannot be represented accurately by a second-order diffusion equation. This phenomenon is called anomalous diffusion, which is characterized by the

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nonlinear growth of the mean square displacement, of a diffusion particle over time. The anomalous diffusions differ according to the values of $\alpha$, where $\alpha$ is the order of the fractional derivative. The fractional diffusion equation is compatible with observations of plumes in the laboratory and the field. It predicts power law, faster than linear scaling of the apparent plume variance. Moreover, the fractional diffusion equation is the governing equation of all 1-D stable random walks. If the walks are not heavy-tailed (i.e., have finite variance), then the classical central limit theorem gives $\alpha = 2$ [2].

With an expanding range of applications and models based on fractional calculus comes a need for the development of robust and accurate computational methods for solving these equations. Publications on the numerical schemes for solving spatial fractional PDEs, are relatively sparse, and the majority of the publications are based on finite difference methods [9,42,45,48,49,72–75]. Some other numerical schemes using low-order finite elements [21,29,36,63], discontinuous Galerkin time stepping method [47], matrix transfer technique [38], point interpolation method [46], dual reciprocity boundary integral equation technique [16], homotopy analysis method [15], and spectral methods [13] have also been proposed.

One of the key issues with fractional diffusion models is the design of efficient numerical schemes for the space and time discretization. Until now, most models have relied on the finite difference method to discretize both the fractional-order space diffusion term [42,48,49,72–74] and time derivative [11,44,61]. Some numerical schemes using low-order finite elements have also been proposed [29,36,63]. Some works providing an introduction to fractional calculus related to diffusion problems are, for instance, [1,32,50,51,71,74,77]. In this work, we will be interested in the superdiffusion, for $1 < \alpha \leq 2$ and experimental evidence of this type of diffusion is already reported in several works [2,37,58,78]. Since fractional derivatives are non-local operators, finite difference and finite element schemes generate large and full coefficient matrices. One approach is to discretize these non-local differential operators with non-local numerical methods. Following that approach, Hanert has proposed a pseudo-spectral method based on Chebyshev basis functions in space and Mittag–Leffler basis functions in time to discretize the space-time fractional diffusion equation [34]. A similar approach has been followed by Li and Xu to discretize the time-fractional diffusion equation with a Jacobi pseudo-spectral method [43]. Recently, Xu and Hesthaven proposed stable multi-domain spectral penalty methods for solving fractional PDEs [76].

Unlike traditional numerical methods for solving differential equations, meshless methods need no mesh generation, which is the major problem in finite difference, finite element and spectral methods [53–56,62,64,70]. Radial basis functions methods are truly meshless and simple enough to allow modelling of rather high dimensional problems [14,18,31,39,56,62,64]. These methods can be very efficient numerical schemes to discretize non-local operators like fractional differential operators. These basis functions can be clustered in a specific region to locally increase the accuracy of the method. On the other hand, the basis functions used in the RBF expansion are high-order functions that span the entire domain like with the pseudo-spectral method. It was shown that RBFs converge to pseudo-spectral methods in their flat radial function limit, making RBFs a generalization to pseudo-spectral methods, for scattered nodes and non-flat radial functions [26]. RBFs have several advantages over spectral/pseudo spectral methods: in addition to offering a flexibility in terms of the shape of the domain, they allow a local node refinement, and an easy generalization to higher dimensions. Recently, Piret and Hanert [60] proposed a Guassian RBF discretization for the one-dimensional space-fractional diffusion equations.

For time-dependent PDEs, meshless kernel-based methods are based on a fixed spatial interpolation, but since the coefficients are time-dependent, one obtains a system of ODEs. This is the well-known *method of lines*, and it turned out to be accurate in several problems [17,23,53].

In this paper, we first provide the required formulas for the Riemann–Liouville fractional derivative of the five kinds of RBFs, including the Powers, Gaussian, Multiquadric, Matérn and Thin-plate splines, in one dimension. Then we consider discretizations of the following fractional diffusion equation with the RBF collocation method:

$$u_t(x,t) - d(x) D^{\alpha}_{0^+} u(x,t) = s(x,t), \qquad x \in (a,b), t \in (0,T], 1 < \alpha \leq 2, \tag{1}$$

with the initial and mixed boundary conditions

$$u(x,0) = \phi(x), \qquad x \in [a,b], \tag{2}$$

$$\beta_1 u(a,t) + \gamma_1 u_x(a,t) = g_1(t), \qquad t \in [0,T], \tag{3}$$

$$\beta_2 u(b,t) + \gamma_2 u_x(b,t) = g_2(t), \qquad t \in [0,T], \tag{4}$$

where the diffusion coefficient (or diffusivity) $d(x) > 0$. The coefficients $\beta_1$, $\beta_2$, $\gamma_1$ and $\gamma_2$ are constant and $s$, $\phi$, $g_1$, and $g_2$ are known functions. The parameter $\alpha$ is the order of Riemann–Liouville fractional derivative, for $x \in [a,b]$, $-\infty \leq a < b \leq \infty$, defined by

$$(D^{\alpha}_{a^+} f)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x f(t)(x-t)^{n-\alpha-1} dt, \qquad (x > a, n = [\alpha] + 1). \tag{5}$$

In the case $\alpha = 2$, Eq. (1) is the classical diffusion equation.

The function $u(x, t)$ under consideration which is solution of (1), should be such that the corresponding integral (5) converges. If the function $u(x, t)$ vanishes at infinity, we have absolute convergence of such integrals for a wide class of functions [65]. However, these functions do not necessarily need to vanish at infinity and we can find under which conditions these