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The long-time L^2 and H^1 stability of linearly extrapolated second-order time-stepping schemes for the 2D incompressible Navier–Stokes equations

Sean Breckling[a,*], Sidney Shields[b]

[a] Department of Applied and Computational Mathematics and Statistics, University of Notre Dame, Notre Dame, IN, United States
[b] Department of Mathematics, Pacific Union College, Angwin, CA, United States

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ABSTRACT

Herein we present a study on the long-time stability of finite element discretizations of a generalized class of semi-implicit second-order time-stepping schemes for the 2D incompressible Navier–Stokes equations. These remarkably efficient schemes require only a single linear solve per time-step through the use of a linearly-extrapolated advective term. Our result develops a class of sufficient conditions such that if external forcing is uniformly bounded in time, velocity solutions are uniformly bounded in time in both the L^2 and H^1 norms. We provide numerical verification of these results. We also demonstrate that divergence-free finite elements are critical for long-time H^1 stability.

1. Introduction

On a two or three-dimensional (d) domain Ω with an initial condition $\mathbf{u}_0 \in L^2(\Omega)^d$, and external force $\mathbf{f}(\mathbf{x}, t) \in L^\infty(\mathbb{R}_+, H^{-1}(\Omega)^d)$, the incompresible Navier–Stokes equations (NSE) are written

$$\mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}, \tag{1.1}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{1.2}$$

where $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ denotes a velocity, $p = p(\mathbf{x}, t)$ is the pressure, and ν represents the kinematic viscosity.

A comprehensive understanding of the long-time behavior of discretizations of the NSE is crucial when simulating flows over extended time-scales. These settings are common in geophysical flows, e.g., weather prediction and atmospheric models. In this article we demonstrate that divergence-free finite element discretizations of the NSE paired with a class of second-order semi-implicit time-stepping schemes are long-time stable in both the L^2 and H^1 norms.

These extrapolated semi-implicit schemes linearize the advective velocity with an Adams-Bashforth-type extrapolated term. This approach is substantially more efficient; preserving the $\mathcal{O}(\Delta t^2)$ asymptotic rate while requiring only a single linear solve per time-step, in contrast to the fully implicit schemes which require fixed-point iterations.

Practitioners have long understood that implicit and semi-implicit time-stepping schemes are well suited for longer time-scale simulations, but under what conditions? While such questions have been studied since the 1980s [1–3], there has been

* Corresponding author.
E-mail addresses: sbreckli@nd.edu (S. Breckling), sshields@puc.edu (S. Shields).

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a strong recent effort to extend these results [4–11]. Tone and Wirosoetisno proved that, under a CFL condition, uniform bounds in the L^2 and H^1 norms exist for both the implicit Euler, and Crank–Nicolson time-stepping schemes [4,5]. Rebholz et al. demonstrate the long-time L^2 stability of finite element discretizations of semi-implicit linearly extrapolated Crank–Nicolson (CNLE) and BDF2 (BDF2LE) time-stepping schemes in [9,11], respectively. In [9] there's a strong case made for the importance of divergence-free finite elements (i.e., finite elements that allow for strong enforcement of (1.2)) in long-time stability results. Another particularly interesting result is seen in [10] where Heister et al. demonstrate that when the 2D incompressible NSE are discretized in the velocity-vorticity-helicity (VVH) formulation using BDF2 time-stepping and divergence-free finite elements that H^1 long-time stability is guaranteed unconditionally.

One of our goals is to further generalize the long-time L^2 stability results seen in [9,11] by developing a result on the so-called θ methods. Suppressing spatial discretization, the θ-methods are written

$$\frac{(\theta+\frac{1}{2})\mathbf{u}^{n+1}-2\theta\mathbf{u}^{n}+(\theta-\frac{1}{2})\mathbf{u}^{n-1}}{\Delta t}-\nu\triangle(\theta\mathbf{u}^{n+1}+(1-\theta)\mathbf{u}^{n})+\Big((1+\theta)\mathbf{u}^{n}-\theta\mathbf{u}^{n-1}\Big)\cdot\nabla\Big(\theta\mathbf{u}^{n+1}+(1-\theta)\mathbf{u}^{n}\Big)$$
$$+\nabla\Big(\theta p^{n+1}+(1-\theta)p^{n-1}\Big)=\mathbf{f}(\mathbf{x},t_{n+\theta}). \tag{1.3}$$

The parameter θ is selected on the interval $[\frac{1}{2},1]$. Finite element discretizations of this scheme were shown to be optimally accurate in [12]. When $\theta=\frac{1}{2}$ or $\theta=1$ the CNLE and BDF2LE schemes are recovered. An affirmative long-time stability result in the L^2 norm is not a surprise, since the CNLE scheme does not dissipate energy in time, compared to the BDF2LE scheme which does. Certainly an energy-dissipating scheme "between" should enjoy similar long-time stability properties.

Our main result addresses long-time stability in the H^1 norm, in 2D, for $\theta\in(\frac{1}{2},1]$. This result guarantees that under a CFL condition, the discrete energy dissipation rate has a uniform upper-estimate in time. We additionally extend the principal result of [9], demonstrating the importance of divergence-free finite elements for long-time H^1 stability. It is a straight-forward matter to confirm

$$\frac{\nu}{2}||\nabla\cdot\mathbf{u}||^2\le\nu||\nabla\mathbf{u}||^2. \tag{1.4}$$

Thus, any non-physical error seen in the divergence guarantees a positive, potentially large, lower estimate for the energy dissipation rate. Such error can potentially destroy the reliability of these schemes, given that its contributions to the discrete energy inequality accumulate over time.

The remainder of this paper is arranged as follows. In Section 2 we establish the appropriate spaces and norms used in our finite element methods, as well as provide shorthand and helpful estimates used in our analysis. In Section 3 we present our finite element method, as well as our analysis for the long-time L^2 and H^1 stability. Results from three numerical experiments are presented in Section 4, and we conclude in Section 5 with a brief summary.

2. Notation and preliminaries

In this section we present the mathematical settings, as well as provide helpful notation for our analysis.

2.1. Continuous and discrete function spaces and norms

Internal flows are typically formulated in the spaces

$$\mathbf{X}:=\{\mathbf{v}\in\big(H^1(\Omega)\big)^d\ :\ \mathbf{v}|_{\partial\Omega}=0\},$$

$$Q:=\Big\{p\in L^2(\Omega)\ :\ \int_\Omega p\ dx=0\Big\},$$

where the domain Ω is an open, connected, bounded set with either a C^2 smooth boundary, or a convex polygon/polyhedron. The set of weakly divergence-free functions in $\mathbf{X}$ are given by

$$\mathbf{V}=\{\mathbf{v}\in\mathbf{X}:(q,\nabla\cdot\mathbf{v})=0,\ \forall q\in Q\}.$$

Norms in the Sobolov spaces $H^k(\Omega)$ are written $||\cdot||_k$. The $L^2(\Omega)$ norm and inner product are denoted by $||\cdot||$ and $(\cdot,\ \cdot)$.

Herein we consider finite element spaces on triangular (2D) and tetrahedral (3D) discretizations such that $\mathbf{X}_h\subset\mathbf{X}$ and $Q_h\subset Q$ satisfy the discrete inf-sup (or *LBB*) condition

$$\inf_{q_h\in Q_h}\sup_{\boldsymbol{\phi}_h\in\mathbf{X}_h}\frac{(q_h,\nabla\cdot\boldsymbol{\phi}_h)}{||\nabla\boldsymbol{\phi}_h||||q_h||}\ge\beta^h>0, \tag{2.1}$$

where β^h is bounded away from zero uniformly in Ω. For element-wise polynomials of degree k, P_k, the Taylor-Hood (TH) (P_k,P_{k-1}) finite elements are *LBB*-stable on quasi-uniform triangular (2D)/tetrahedral (3D) meshes [13]. The (P_k,P_{k-1}^{disc}) Scott–Vogelius finite elements (SV) allow for discontinuities between elements in the pressure space. This element choice guarantees $\nabla\cdot\mathbf{X}_h\subset Q_h$, and allows for the discrete divergence-free condition to be strongly enforced, i.e., $\mathbf{V}_h\subset\mathbf{V}$, where

$$\mathbf{V}_h=\{\mathbf{v}_h\in\mathbf{X}_h:(\nabla\cdot\mathbf{v}_h,q_h)=0\ \forall q_h\in Q_h\}.$$