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Necessary gradient restrictions at the core of a voting rule ☆

John Duggan *

Department of Political Science, University of Rochester, United States
Department of Economics, University of Rochester, United States

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ABSTRACT

This paper generalizes known gradient restrictions for the core of a voting rule parameterized by an arbitrary quota. For the special case of majority rule with an even number of voters, the result implies that given any pointed, finitely generated, convex cone C, the difference between the number of voters with gradients in C and the number with gradients in $-C$ cannot exceed the number of voters with zero gradient, plus a dimensional adjustment. When the cone has dimensionality less than three, the adjustment is zero. A difficulty in the proof of a result of Schofield (1983), which neglects the dimensional adjustment term, is identified, and a counterexample (in three dimensions) presented.

1. Introduction

In the spatial theory of voting, a collection of voters must choose an alternative from a subset of Euclidean space. Assuming voting is governed by a quota rule, whereby one alternative defeats another in a pairwise vote whenever the number of supporters meets the quota, the set of alternatives that are stable with respect to pairwise voting is the *core* of the voting rule. Under standard differentiability conditions, it has been known since Plott (1967) that the conditions defining the majority core for an odd number of voters imply restrictions on voter gradients, and that when the set of alternatives is multidimensional, these restrictions are severe. Specifically, at any majority core alternative, there must be at least one voter whose gradient equals zero; and if there are no other such voters, then the gradients of the remaining voters must satisfy *radial symmetry:* they can be paired in such a way that the voters in each pair have gradients that point in exactly opposite directions. For majority voting with an even number of voters, or for general quota rules, our knowledge of the gradient restrictions that must hold at the core is incomplete. This paper provides a necessary gradient condition for the core of a quota rule that generalizes known results in the literature, and it identifies a difficulty with the proof of – and presents a counterexample to – a result by Schofield (1983) for the majority core with an even number of voters.

The main result of this paper states that at any core alternative, for every well-behaved cone C,[1] the following must be less than the quota: (i) the number of voters whose gradients belong to C, plus (ii) half of the number of voters whose gradients do not lie on the linear subspace spanned by C, plus (iii) $\frac{|I|}{s}$. Here, s is the dimensionality of the cone, and I is the coalition of voters with gradients in the subspace spanned by the cone, less the voters whose gradients belong to C or $-C$. This last term is a "dimensional adjustment", which derives from the difficulty in securing votes from the members of I, while at the same time garnering the support of voters with gradients in C. The result has numerous implications, including results of Matthews (1980), Banks (1995), and Saari (1997).[2] Saari's (1997) Lemma 2, for example, establishes that if an alternative x belongs to the core of the voting rule with quota q, and if n denotes the number of voters and k the number of voters with zero gradient at x, then for each coalition G with at least $2q-n+k-1$ members with non-zero gradients, the linear subspace spanned by the gradients of voters in G must contain a collection of voter gradients that is semi-positively dependent, i.e., there must be a coalition G' with gradients in this subspace such that the zero vector is a convex combination of gradients of the voters in G'. Such a restriction is "linear", because it yields existence of a voter i whose gradient is a linear combination of the gradients of the members of G.

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* Correspondence to: Wallis Institute of Political Economy, University of Rochester, Rochester, NY 14627, United States.
E-mail address: dugg@ur.rochester.edu.

1 That is, for every pointed, finitely generated, convex cone.

2 Assuming a weighted majority rule, McKelvey et al. (1980) provide a gradient restriction for a core alternative such that the set of voters for whom this alternative is ideal has measure zero; in a model with a finite number of voters, this means that no voter's ideal point is located at the core. McKelvey and Schofield (1987) derive gradient restrictions for voting rules generated from a collection of decisive coalitions. For the special case of a quota rule, the "pivotal gradient condition" of the latter authors reduces to the necessary condition provided by Matthews' (1980) Theorem 1.

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A stronger type of restriction would be "conic", in the sense that it identifies a voter i whose gradient belongs to $-C$, where C is the cone generated by the gradients of members of G. The extant literature does not contain conic gradient restrictions for quota rules, but a corollary of the main result presented here is that if the dimensionality of a well-behaved cone C is $s \geq 2$ and the size of G exceeds $\frac{sq-n+k-s}{s-1}$, then there is a voter whose gradient belongs to $-C$. Interestingly, Schofield's (1983) Lemma 1 states a conic restriction for the majority core: given any well-behaved cone C, the difference between the number of voters with gradients in C and the number with gradients in $-C$ cannot exceed the number of voters with zero gradient. However, as demonstrated in a counterexample, this result fails due to the absence of the dimensional adjustment mentioned above. Another corollary of our main result is that if the cone C has dimension equal to one or two, then Schofield's inequality does indeed hold. A further implication is that at a majority core alternative x, for every pair of voters whose gradients do not point in opposite directions, there must be another voter such that the three voters' gradients are semi-positively dependent. This pairwise conic restriction is used by Chung and Duggan (2018) in the analysis of their concept of directional equilibrium for an even number of voters: Theorem 1 of that paper shows that at a majority core alternative x such that one voter has gradient equal to zero (so radial symmetry need not hold), the norm of the sum of normalized gradients at x is less than or equal one.

Section 2 presents the model of spatial voting, and Section 3 provides a detailed review of gradient restrictions in the literature. Section 4 presents the main result and illustrates how previous results are obtained as special cases. Section 5 derives several new results as implications of the main result. Section 6 presents the counterexample to the conic restriction of Schofield (1983). Section 7 ends with a discussion of the role of differentiability in the results.

2. Spatial voting model

Let $N = \{1, \ldots, n\}$ be a set of voters, and let $X \subseteq \mathbb{R}^d$ be a nonempty set of alternatives, modeled as a subset of d-dimensional Euclidean space. Assume that the preferences of voter i are represented by a continuously differentiable utility function $u_i : X \to \mathbb{R}$. For use in examples, say the utility function of voter i is *Euclidean* if she prefers alternatives that are closer to her ideal alternative, i.e., there is an *ideal point* $\hat{x}^i \in X$ such that for all $x, y \in X$, we have $u_i(x) > u_i(y)$ if and only if $\|x - \hat{x}^i\| < \|y - \hat{x}^i\|$. In this case, we can assume quadratic utility, i.e., $u_i(x) = -\|x - \hat{x}^i\|^2$, without loss of generality. The necessary conditions derived in the paper only consider marginal changes in any direction from a given alternative, and as a consequence, global assumptions such as quasi-concavity of utilities are not used.

For simplicity, we write p^i_x for the normalized gradient of voter i's utility function evaluated at x, formally defined as follows: if $\nabla u_i(x) \neq 0$, then we set

$$p^i_x = \frac{1}{\|\nabla u_i(x)\|} \nabla u_i(x),$$

and if $\nabla u_i(x) = 0$, then we set $p^i_x = 0$. Given any quota q satisfying $\frac{n}{2} < q \leq n$, the *q-voting rule* is the binary relation $\succ_q$ on X defined as follows: for all $x, y \in X$, $x \succ_q y$ if and only if $|\{i \in N : u_i(x) > u_i(y)\}| \geq q$. Then the *q-core* is the set of maximal elements of $\succ_q$, i.e., it is

$$C(q) = \{x \in X : \text{ there does not exist } y \in X \text{ such that } y \succ_q x\}.$$

Setting $q^m = \lceil \frac{n+1}{2} \rceil$, we obtain majority rule as a special case,[3] and we refer to the q^m-core simply as the *majority core*. If $q > q^m$, then we refer to $\succ_q$ as a *supermajority rule*.

[3] The operation $\lceil \cdot \rceil$ denotes integer ceiling.

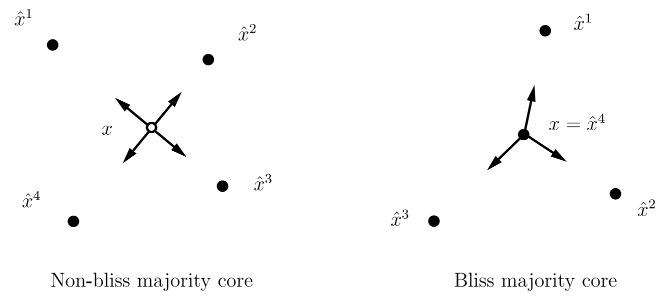

Fig. 1. Majority core, $n = 4$.

It is well-known that the maximality condition in the definition of the majority core implies restrictions on the gradients of voters at a core alternative, and that in a multidimensional space of alternatives, these restrictions can be quite strong. Plott (1967) shows that for majority rule with n odd, if $x \in \text{int}X$ belongs to the majority core, then there must be at least one voter i with zero gradient at x, i.e., $p^i_x = 0$; moreover, if there are no other voters with zero gradient at x, then the gradients of the remaining voters must satisfy *radial symmetry,* in the sense that for every direction t,[4] the number of voters with gradients pointing in the t direction must equal the number with gradients pointing in the $-t$ direction, i.e.,

$$|\{i \in N : p^i_x = t\}| = |\{i \in N : p^i_x = -t\}|. \qquad (1)$$

In the literature, these necessary conditions for majority rule with an odd number of voters are known as the *Plott conditions.*

For other voting rules, including majority rule with n even, gradient restrictions are not as sharp; in particular, there may or may not be voters with zero gradient at a core alternative, and the restrictions on others' gradients will depend on that contingency. Following Saari (1997), we define the *bliss q-core* to consist of every q-core alternative x such that for some voter i, we have $p^i_x = 0$; and we define the *non-bliss q-core* to consist of every q-core alternative such that for all voters i, we have $p^i_x \neq 0$. Assuming n is even, it is known that if $x \in \text{int}X$ is a non-bliss majority core alternative, then the radial symmetry condition in (1) must hold at x. If $x \in \text{int}X$ is a bliss majority core alternative, then there must of course be some voter i with $p^i_x = 0$, but radial symmetry need not hold; see Fig. 1, which depicts four voters with Euclidean preferences and two arrangements of ideal points, one in which x belongs to the non-bliss majority core, and one in which x belongs to the bliss majority core.

3. Review of gradient restrictions at the core

Beyond majority rule, the literature on the theory of voting has provided a number of gradient restrictions for supermajority voting rules. Smale (1973) proves that if an alternative $x \in \text{int}X$ is Pareto optimal for a coalition G of voters, then we must have $0 \in \text{conv}\{p^i_x : i \in G\}$; indeed, if this did not hold, then the separating hyperplane theorem could be applied to deduce a direction t such that $p^i_x \cdot t > 0$ for all $i \in G$, but then we could define $y = x + \epsilon t \in X$ for $\epsilon > 0$ sufficiently small that $u_i(y) > u_i(x)$ for all $i \in G$, contradicting the assumption that x is Pareto optimal for G. For a general quota rule, Banks (1995) uses this result to argue that if x belongs to the q-core, then for every coalition G of voters with $|G| \geq q$, we have $0 \in \text{conv}\{p^i_x : i \in G\}$, and he then establishes generic emptiness of the q-core when the dimensionality d of the set of alternatives is sufficiently large.

[4] The term *direction* refers to any vector in $\Re^d$ with norm one.