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# Tests of stochastic monotonicity with improved power

Juwon Seo

*Department of Economics, National University of Singapore, Singapore*

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We develop improved statistical procedures for testing stochastic monotonicity. While existing tests use a fixed critical value to set the limiting rejection rate equal to nominal size at the least favorable case, we use a bootstrap procedure to raise the limiting rejection rate to nominal size over much of the null. This improves power against relevant local alternatives. To show the validity of our approach we draw on recent results on the directional differentiability of the least concave majorant operator, and on bootstrap inference when smoothness conditions sufficient to apply the functional delta method for the bootstrap are not satisfied.



## 1. Introduction

In stochastic modeling, a variety of orderings can be used to compare the 'magnitude' of random variables, such as stochastic dominance, mean residual life ordering, likelihood ratio ordering, positive dependence ordering, and so on. In this paper, we focus on *stochastic monotonicity*, an ordering of random variables based on the first order stochastic dominance of conditional distributions: for two random variables $X$ and $Y$, with $F_{Y|X}(\cdot|x)$ denoting the cumulative distribution function of $Y$ conditional on $X = x$, we say *$Y$ is stochastically increasing in $X$* (or equivalently, *$Y$ is positive regression dependent on $X$*) if and only if $F_{Y|X}(y|x)$ is a nonincreasing function of $x$ for all $y$. In what follows, we denote by $\mathcal{X}$ and $\mathcal{Y}$ the supports of $X$ and $Y$ respectively, and consider the conditional distribution $F_{Y|X}$ on $\mathcal{Y} \times \mathcal{X}$. This paper studies statistical methods to test stochastic monotonicity with the null hypothesis of $Y$ being stochastically increasing in $X$.

As a natural way to examine the monotonic relationship of random variables, stochastic monotonicity can be of interest in many applications. Suppose for example, we examine empirically whether a son's social status is determined by that of his parents. This is a question about intergenerational mobility, one of the classic subjects in sociology and labor economics (Becker and Tomes, 1979, 1986; Mulligan, 1999; Han and Mulligan, 2001; Restuccia and Urrutia, 2004). The conventional approach to the problem has been to investigate the dependence between son's and parents' status measured by wage, for instance, and verify that their incomes have positive correlation or positive quadrant dependence. Stochastic monotonicity, implying both positive correlation and positive quadrant dependence, can provide more information on this aspect of intergenerational mobility because it also identifies nonlinear or nonmonotone aspects of the relationship between son's income and parent's income that would be undetectable with a test of positive correlation or positive quadrant dependence. For instance, if a larger portion of the children of very wealthy parents has a high probability of earning very low income than the children of moderately wealthy parents, perhaps due to perverse incentives arising from the anticipation of

*E-mail address:* ecssj@nus.edu.sg.

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inheritance, then this would violate stochastic monotonicity but can be consistent with positive correlation or positive quadrant dependence.[1]

In testing stochastic monotonicity, the existing statistical methods rely on procedures that use the critical values to control the limiting rejection rates at the least favorable case (hereafter lfc), i.e., a point in the null at which the asymptotic distribution of the test statistic is largest in the sense of stochastic dominance. Lee et al. (2009) were the first to propose a test for stochastic monotonicity, using a kernel smoothed U-statistic to assess the monotonicity of the conditional distribution in the conditioning variable. Later, Delgado and Escanciano (2012) have suggested a test based on the distance between the empirical copula and its least concave majorant. In both tests, the asymptotic null distributions are derived under the independence of $X$ and $Y$ which turns out to be the lfc. As a consequence, the tests are conservative at all points in the null but the least favorable case. This suggests that power against relevant local alternatives may be limited.

In this paper we show how to improve the power of tests for stochastic monotonicity by using a modified bootstrap technique to raise the limiting rejection rate of the test to the nominal significance level over a wide region of the null hypothesis. More specifically, we first derive the limit distributions of a generalization of the statistic of Delgado and Escanciano (2012) at all points in the null. We then discuss how the standard bootstrap fails to apply, and propose an alternative bootstrap method. To show the validity of our approach we draw on recent results on the directional differentiability of the least concave majorant operator (Beare and Moon, 2015; Beare and Shi, 2017) and on the application of the functional delta method with directionally differentiable operators (Fang and Santos, 2016).

## 2. Null hypothesis and test statistic

Let $X$ and $Y$ be continuous random variables, and let $C(u, v)$ denote the copula of $Y$ and $X$. The null hypothesis of stochastic monotonicity can be reformulated in terms of the shape of this copula function. Theorem 5.2.10 and Corollary 5.2.11 in Nelsen (2006) state that $Y$ is stochastically increasing in $X$ if and only if $C(u, v)$ is concave in $v$ for any $u \in [0, 1]$. We shall therefore write our null hypothesis as $H_0 : C \in \Theta_0$, where

$$\Theta_0 = \{C \in \Theta : C(u, \cdot) \text{ is concave for each fixed } u \in [0, 1]\},$$

with $\Theta$ denoting the collection of bivariate copula functions on $[0, 1]^2$ with continuous partial derivatives. The alternative hypothesis is $H_1 : C \in \Theta_1$, where $\Theta_1 = \Theta \setminus \Theta_0$.

It is clear that the partial concavity introduced in the preceding paragraph is not as strong as the general notion of concavity of a bivariate function. While it is well known that the only concave copula is the Fréchet–Hoeffding upper bound $C(u, v) = \min(u, v)$, there are many copulas which have concave vertical sections,[2] and thus belong to $\Theta_0$. Vertical sections of copulas, in fact, can be any functions that are nondecreasing and 1-Lipschitz, provided that they stay between the Fréchet–Hoeffding upper and lower bounds. Thus, they may be concave, convex, or otherwise. In Table 2.1, we provide the conditions for some parametric copulas to be in $\Theta_0$.[3]

Having clarified the null hypothesis, we shall now proceed to the construction of our test statistics. Suppose we observe $n$ independent and identically distributed copies of $(X, Y)$, denoted by $(X_i, Y_i)$, $i = 1, \ldots, n$. Define the empirical cdfs of $X$ and $Y$, and the empirical copula of $Y$ and $X$ as

$$F_{X,n}(x) = \frac{1}{n}\sum_{i=1}^{n} 1(X_i \le x), \quad F_{Y,n}(y) = \frac{1}{n}\sum_{i=1}^{n} 1(Y_i \le y) \quad \text{for } (x, y) \in \mathbb{R}^2,$$

$$C_n(u, v) = \frac{1}{n}\sum_{i=1}^{n} 1\{F_{Y,n}(Y_i) \le u, F_{X,n}(X_i) \le v\} \quad \text{for } (u, v) \in [0, 1]^2. \tag{2.1}$$

Our test statistic is of the form

$$M_n^p = n^{1/2} \left\| \tilde{\mathcal{M}} C_n - C_n \right\|_p \tag{2.2}$$

where $\| \cdot \|_p$ is the $L^p$-norm with respect to the Lebesgue measure on $[0, 1]^2$ given $p \in [1, \infty]$, and $\tilde{\mathcal{M}}$ is the partial least concave majorant (hereby, partial lcm) operator applied to the second argument.[4] In order to provide a formal definition of $\tilde{\mathcal{M}}$, we shall begin by reviewing the definition of the least concave majorant (lcm) operator $\mathcal{M}$, and also of the restricted lcm

[1] Such questions about the monotonicity arise naturally in many fields of economics. Milgrom (1981) and Milgrom and Shannon (1994) have a general discussion on the central role of monotonicity in classical economic theory. In particular, stochastic monotonicity is one of the key conditions for certain Markovian models to have a stationary distribution in Lucas and Stokey (1989) and Hopenhayn and Prescott (1992). In the IV literature, Blundell et al. (2007) assume stochastic monotonicity to release the exclusion restriction, while Small et al. (2014) introduce stochastic monotonicity to relax monotonicity restrictions of IV. We refer to the introductory part of Lee et al. (2009) and Delgado and Escanciano (2012) for more motivation.

[2] The vertical section of $C$ at $u = u_0$ refers to the cross-section of the copula at a point $u_0 \in [0, 1]$, i.e., $C(u_0, v)$.

[3] Following convention, $\Phi^{-1}$ is the quantile function of the standard normal distribution and $N_\rho$ is the joint cumulative normal distribution function with mean zero and correlation $\rho$. $t_\nu^{-1}$ denotes the inverse cumulative distribution of the univariate $t$-distribution with degree of freedom $\nu$ and $t_{\rho,\nu}$ is the multivariate $t$-distribution with degree of freedom $\nu$, scale parameter $\rho$ and location parameter zero.

[4] When $p = \infty$, $M_n^\infty$ corresponds to the test statistic in Delgado and Escanciano (2012).

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