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Non computable Mandelbrot-like sets for a one-parameter complex family

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ABSTRACT

We show the existence of computable complex numbers λ for which the bifurcation locus of the one-parameter complex family $f_b(z) = \lambda z + bz^2 + z^3$ is not Turing computable.

1. Introduction

The Mandelbrot set $\mathcal{M}$ is defined as the set of complex parameters c for which the orbit of 0 under iteration of the quadratic map $P_c(z) = z^2 + c$:

$$0, P_c(0), P_c^2(0), P_c^3(0), \ldots$$

remains bounded. $\mathcal{M}$ is widely known for the spectacular beauty of its fractal structure, and an enormous amount of effort has been made in order to understand its topological and geometrical properties (Fig. 1). This effort has greatly relied on computer simulations, and it is most natural to ask *whether these simulations can be trusted*. A form of this question was first asked by Penrose in [1] and has been a subject of much interest. Its answer, however, remains unknown.

For a complex polynomial map P, let us recall that the *filled Julia set* $K(P)$ corresponds to the set of points $z \in \mathbb{C}$ whose orbit under iterations by P remains bounded, and that the *Julia set* $J(P)$ is defined as the boundary of $K(P)$. The family $P_c(z) = z^2 + c$, with $c \in \mathbb{C}$, is known as the *quadratic family*, and the Mandelbrot set $\mathcal{M}$ can be equivalently defined as the set of complex parameters c for which the Julia set $J(P_c)$ is connected. As such, it is often referred to as the *connectedness locus* of the quadratic family. The boundary of $\mathcal{M}$ corresponds to the parameters near which the geometry of the Julia set undergoes a dramatic change. For this reason, its boundary $\partial\mathcal{M}$ is referred to as the *bifurcation locus*.

The central open conjecture in complex dynamics is known as Density of Hyperbolicity Conjecture. This conjecture is widely expected to be true, and postulates that $\mathcal{M}$ is the closure of the open set of parameter values c for which P_c exhibits *hyperbolic dynamics*. The latter simply means that $|DP_c^n(z)| > 1$ on a neighborhood of $J(P_c)$ for some $n \in \mathbb{N}$. In [2],

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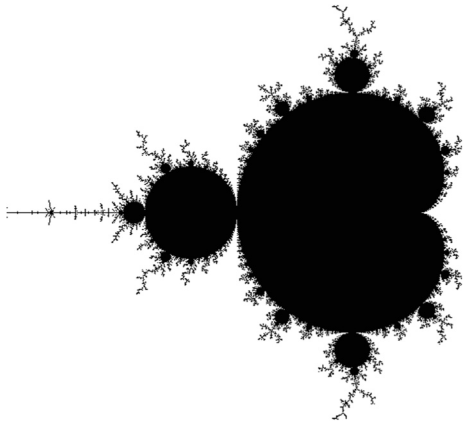

Fig. 1. The Mandelbrot set $\mathcal{M} = \{c \in \mathbb{C} : \sup_n |P_c^n(0)| < \infty\}$.

Hertling demonstrated that Density of Hyperbolicity Conjecture implies that the Mandelbrot set $\mathcal{M}$, as well as its boundary, the bifurcation locus $\partial\mathcal{M}$, are rigorously computable. By definition, a compact set is *computable* if we can visualize it on a computer screen and zoom-in with an arbitrarily high magnification (see Section 2).

In this paper we show that such a computability property cannot be taken for granted. We consider a different one-parameter family of complex dynamical systems, studied by Buff and Henriksen in [3]:

$$f_b(z) = \lambda z + bz^2 + z^3, \qquad b \in \mathbb{C}, \tag{1}$$

where $\lambda \in \mathbb{C}$ satisfies $|\lambda| = 1$. We denote by $\mathcal{M}_\lambda$ the connectedness locus of the family, that is, the set of complex parameters b for which the Julia set $J(f_b)$ is connected.

Our main result is the following.

Main Theorem. *There exists a computable (by an explicit algorithm) value of λ such that the bifurcation locus $\partial\mathcal{M}_\lambda$ is not computable.*

We remark that our proof strategy cannot be applied to the quadratic family in order to obtain a similar statement for the Mandelbrot set $\mathcal{M}$. As we will explain below, our approach requires the presence of a persistent indifferent fixed point, and the family given by (1) is precisely the simplest possible one-parameter family with this property.

1.1. Main technical contributions and proof of the Main Theorem

A principal result of [3] is that for each λ of modulus 1, the bifurcation locus $\partial\mathcal{M}_\lambda$ contains homeomorphic[5] copies of the quadratic Julia set $J(\lambda z + z^2)$ (see Fig. 2 for an illustration). Our proof of the Main Theorem relies on a computable version of this statement combined with the fact that the Julia set $J(\lambda z + z^2)$ *is not computable* for certain computable values of λ [4] (see also Theorem 2.3).

In order to carry out this strategy, we develop computable versions of a variety of central tools in complex dynamics, such as *external rays, straightening maps* and *holomorphic motions*. These tools represent the main technical contribution of this paper and we expect them to be of an independent interest. In the remainder of this section, we briefly outline the main principles underlying these techniques and explain how they are used to obtain our Main Theorem.

Let $\lambda \in \mathbb{C}$ of modulus 1 be fixed. One starts by choosing a reference parameter b_1 (see Section 3.2) and considering the associated map $f_{b_1}(z) = \lambda z + b_1 z^2 + z^3$ in (1). One then considers

$$Q_{b_1} \equiv f_{b_1}|_{U'}$$

the restriction of f_{b_1} to a certain open connected set U' containing part of the filled Julia set $K(f_{b_1})$ (see Fig. 3 for an illustration and Section 3.2 for the definition). The filled Julia set $K(Q_{b_1})$ of Q_{b_1} is defined as the set of $z \in U'$ that do not escape from U' under iterations of Q_{b_1}, and its Julia set $J(Q_{b_1}) = \partial K(Q_{b_1})$. It turns out that the restricted map Q_{b_1} behaves in many ways like the quadratic map $\widehat{P}_\lambda(z) = \lambda z + z^2$. In fact, Buff and Henriksen in [3] show that (see Theorem 3.2) there exists an homeomorphism $\phi : J(Q_{b_1}) \to J(\widehat{P}_\lambda)$ conjugating the maps Q_{b_1} and $\widehat{P}_\lambda$ on their Julia sets, that is, such that

$$\phi \circ Q_{b_1}(z) = \widehat{P}_\lambda \circ \phi.$$

[5] In fact, $\partial\mathcal{M}_\lambda$ contains quasi-conformal copies of $J(\lambda z + z^2)$, which roughly means that the underlying map preserves angles up to a multiplicative constant.