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Star-factors in graphs with large minimum degree

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A R T I C L E I N F O

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We prove that every graph of minimum degree d contains a spanning forest in which every component is a star of size at least $\sqrt{d} - \tilde{O}(d^{1/4})$. This improves a result of Alon and Wormald and is optimal up to the lower order term.

1. Introduction

Given graphs G and H, a collection of pairwise vertex-disjoint copies of H in G (not necessarily induced) is called an *H-packing*. This notion generalises matchings (i.e. $H = K_2$) to arbitrary graphs. A *perfect* H-packing (or an *H-factor*) is a packing which covers all the vertices of G. Starting with the seminal paper of Corrádi and Hajnal [4] and its extension by Hajnal and Szemerédi [5], the problem of determining the minimum degree condition which suffices for the existence of an H-factor has received considerable attention. After a series of papers ([2,3,7,8] to name a few), this line of research has culminated with the work of Kühn and Osthus [9] who resolved the problem for every H up to an additive constant term.

Instead of asking for copies of the same graph H, one can ask for copies of graphs from some family $\mathcal{H}$. More precisely, a collection of vertex-disjoint subgraphs $H_1, \dots, H_t \subseteq G$ is an *$\mathcal{H}$-packing* if $H_i \in \mathcal{H}$ for every i. An *$\mathcal{H}$-factor* is defined analogously. In this paper

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we are interested in the family of large *stars*. A star of size ℓ is a complete bipartite graph with ℓ vertices on one side (called *leaves*) and a single vertex on the other (called the *centre*). Let $\mathcal{S}_\ell$ denote the family of all stars of size at least ℓ. In a connection with an analysis of certain exponential time algorithms (see [6]), Alon and Wormald [1] considered the problem of estimating the largest $\ell = \ell(d)$ such that every graph G of minimum degree d contains an $\mathcal{S}_\ell$-factor. Their main result states that $\ell = \Omega((d/\log d)^{1/3})$. Here we improve upon this bound.

Theorem 1.1. *There exist $C \in \mathbb{N}$ such that every graph G with minimum degree d contains an $\mathcal{S}_\ell$-factor for some $\ell \geq \sqrt{d} - Cd^{1/4}\sqrt{\log d}$.*

We make no effort to optimise C. The proof of Theorem 1.1 gives a randomized polynomial time algorithm which constructs such an star-factor. The next theorem shows that the bound on ℓ is almost the best possible without further assumptions on the graph G.

Theorem 1.2. *For every $d \in \mathbb{N}$ and $n \in \mathbb{N}$ sufficiently large there exists a graph G with $n(1 + \lceil\sqrt{d}\rceil) + d$ vertices and minimum degree d which does not contain an $\mathcal{S}_\ell$-factor for any $\ell > \lceil\sqrt{d}\rceil + 1$.*

The next section sets out the tools used in the proof of Theorem 1.1, which is presented in Section 3. The construction from Theorem 1.2 is given in Section 4. Whenever the use of floors and ceilings is not important it will be omitted.

2. Preliminaries

Similarly to other proofs in this line of research, our main ingredient is the Lovász Local Lemma (see [11]). For our purposes its simplest form suffices.

Lemma 2.1 *(Lovász Local Lemma). Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be events in an arbitrary probability space. Suppose that each event $\mathcal{A}_i$ is mutually independent of the set of all the other events but at most d, and $\Pr[\mathcal{A}_i] \leq p$ for all $1 \leq i \leq n$. If*

$$ep(d+1) \leq 1$$

then $\Pr[\bigwedge_{i=1}^n \overline{\mathcal{A}_i}] > 0$.

An algorithmic version of the Local Lemma was given in [10].

The following claim follows easily from Hall's matching criteria, thus we omit the proof.

Claim 2.2. *Let $G = (V_1 \cup V_2, E)$ be a bipartite graph such that $\deg(v) \geq d_1$ for every $v \in V_1$ and $1 \leq \deg(w) \leq d_2$ for every $w \in V_2$, for some $d_1 \geq d_2$. Then G contains an $\mathcal{S}_\ell$-factor for $\ell = \lfloor d_1/d_2 \rfloor$ with all centres being in V_1.*

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