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Non-reconstructible locally finite graphs

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ABSTRACT

Two graphs G and H are *hypomorphic* if there exists a bijection $\varphi\colon V(G) \to V(H)$ such that $G - v \cong H - \varphi(v)$ for each $v \in V(G)$. A graph G is *reconstructible* if $H \cong G$ for all H hypomorphic to G.
Nash-Williams proved that all locally finite connected graphs with a finite number $\geqslant 2$ of ends are reconstructible, and asked whether locally finite connected graphs with one end or countably many ends are also reconstructible.
In this paper we construct non-reconstructible connected graphs of bounded maximum degree with one and countably many ends respectively, answering the two questions of Nash-Williams about the reconstruction of locally finite graphs in the negative.

1. Introduction

Two graphs G and H are *hypomorphic* if there exists a bijection φ between their vertex sets such that the induced subgraphs $G - v$ and $H - \varphi(v)$ are isomorphic for each vertex v of G. We say that a graph G is *reconstructible* if $H \cong G$ for every H hypomorphic

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to G. The *Reconstruction Conjecture*, a famous unsolved problem attributed to Kelly and Ulam, suggests that every finite graph with at least three vertices is reconstructible.

For an overview of results towards the Reconstruction Conjecture for finite graphs see the survey of Bondy and Hemminger [3]. The corresponding reconstruction problem for infinite graphs is false: the countable regular tree T_∞, and two disjoint copies of it (written as $T_\infty \cup T_\infty$) are easily seen to be a pair of hypomorphic graphs which are not isomorphic. This example, however, contains vertices of infinite degree. Regarding locally finite graphs, Harary, Schwenk and Scott [7] showed that there exists a non-reconstructible locally finite forest. However, they conjectured that the Reconstruction Conjecture should hold for locally finite trees. This conjecture has been verified for locally finite trees with at most countably many ends in a series of paper [1,2,11]. However, very recently, the present authors have constructed a counterexample to the conjecture of Harary, Schwenk and Scott.

Theorem 1.1 *(Bowler, Erde, Heinig, Lehner, Pitz [4]). There exists a non-recon-structible tree of maximum degree three.*

The Reconstruction Conjecture has also been considered for general locally finite graphs. Nash-Williams [8] showed that if $p \geqslant 3$ is an integer, then any locally finite connected graph with exactly p ends is reconstructible; and in [10] he showed the same is true for $p = 2$. The case $p = 2$ is significantly more difficult. Broadly speaking this is because every graph with $p \geqslant 3$ ends has some identifiable finite 'centre', from which the ends can be thought of as branching out. A two-ended graph however can be structured like a double ray, without an identifiable 'centre'.

The case of 1-ended graphs is even harder, and the following problems from a survey of Nash-Williams [9], which would generalise the corresponding results established for trees, have remained open.

Problem 1 *(Nash-Williams).* Is every locally finite connected graph with exactly one end reconstructible?

Problem 2 *(Nash-Williams).* Is every locally finite connected graph with countably many ends reconstructible?

In this paper, we extend our methods from [4] to construct examples showing that both of Nash-Williams' questions have negative answers. Our examples will not only be locally finite, but in fact have bounded degree.

Theorem 1.2. *There is a connected one-ended non-reconstructible graph with bounded maximum degree.*

Theorem 1.3. *There is a connected countably-ended non-reconstructible graph with bounded maximum degree.*