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Hamilton cycles in hypergraphs below the Dirac threshold

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ABSTRACT

We establish a precise characterisation of 4-uniform hypergraphs with minimum codegree close to $n/2$ which contain a Hamilton 2-cycle. As an immediate corollary we identify the exact Dirac threshold for Hamilton 2-cycles in 4-uniform hypergraphs. Moreover, by derandomising the proof of our characterisation we provide a polynomial-time algorithm which, given a 4-uniform hypergraph H with minimum codegree close to $n/2$, either finds a Hamilton 2-cycle in H or provides a certificate that no such cycle exists. This surprising result stands in contrast to the graph setting, in which below the Dirac threshold it is NP-hard to determine if a graph is Hamiltonian. We also consider tight Hamilton cycles in k-uniform hypergraphs H for $k \geq 3$, giving a series of reductions to show that it is NP-hard to determine whether a k-uniform hypergraph H with minimum degree $\delta(H) \geq \frac{1}{2}|V(H)| - O(1)$ contains a tight Hamilton cycle. It is therefore unlikely that a similar characterisation can be obtained for tight Hamilton cycles.

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1. Introduction

The existence of Hamilton cycles in graphs is a fundamental problem of graph theory which has been an active area of research for many years. The decision problem – given a graph G, determine if it contains a Hamilton cycle – was one of Karp's famous 21 NP-complete problems [20]. This means we are unlikely to find a 'nice' characterisation of Hamiltonian graphs analogous to Hall's Marriage Theorem and Edmonds's algorithm for the existence of a perfect matching in graphs. Consequently, much research has focussed on sufficient conditions which ensure the existence of a Hamilton cycle in a graph G, such as the classic theorem of Dirac [7] that every graph on $n \geq 3$ vertices with minimum degree at least $n/2$ contains a Hamilton cycle.

In recent years a great deal of attention has been devoted towards establishing analogous results for Hamilton cycles in hypergraphs. To discuss this work we make the following standard definitions.

A *k-uniform hypergraph*, or *k-graph* H consists of a set of vertices $V(H)$ and a set of edges $E(H)$, where each edge consists of k vertices. This generalises the notion of a (simple) graph, which coincides with the case $k = 2$. Given any integer $1 \leq \ell < k$, we say that a k-graph C is an *ℓ-cycle* if C has no isolated vertices and the vertices of C may be cyclically ordered in such a way that every edge of C consists of k consecutive vertices and each edge intersects the subsequent edge (in the natural ordering of the edges) in precisely ℓ vertices. It follows from the latter condition that the number of vertices of an ℓ-cycle k-graph C is divisible by $k - \ell$, as each edge of C contains exactly $k - \ell$ vertices which are not contained in the previous edge. We say that a k-graph H on n vertices contains a *Hamilton ℓ-cycle* if it contains an n-vertex ℓ-cycle as a subgraph; as above, a necessary condition for this is that $k - \ell$ divides n, and we assume this implicitly throughout the following discussion. It is common to refer to $(k - 1)$-cycles as *tight cycles* and to speak of *tight Hamilton cycles* accordingly. This is the most prevalently used definition of a cycle in a uniform hypergraph, but more general definitions, such as a Berge cycle [3], have also been considered. Given a k-graph H and a set $S \subseteq V(H)$, the *degree* of S, denoted $d_H(S)$ (or $d(S)$ when H is clear from the context), is the number of edges of H which contain S as a subset. The *minimum codegree* of H, denoted $\delta(H)$, is the minimum of $d(S)$ taken over all sets of $k-1$ vertices of H, and the *maximum codegree* of H, denoted $\Delta(H)$, is the maximum of $d(S)$ taken over all sets of $k - 1$ vertices of H. Note that for graphs the maximum and minimum codegree are simply the maximum and minimum degree respectively.

1.1. Previous work

The study of Hamilton cycles in hypergraphs has been a thriving area of research in recent years. We briefly summarise some of this work here; for a more expository presentation we refer the reader to the recent surveys of Kühn and Osthus [28], Rödl and Ruciński [30] and Zhao [38]. A major focus has been to find hypergraph analogues of