Journal of Combinatorial Theory, Series B 133 (2018) 211–242

Contents lists available at ScienceDirect

Journal of Combinatorial Theory, Series B

www.elsevier.com/locate/jctb

Counting independent sets in cubic graphs of given girth

Guillem Perarnau, Will Perkins [1]

University of Birmingham, United Kingdom of Great Britain and Northern Ireland

A R T I C L E   I N F O

Article history:
Received 4 November 2016

Keywords:
Independent sets
Independence polynomial
Hard-core model
Petersen graph
Heawood graph
Occupancy fraction

A B S T R A C T

We prove a tight upper bound on the independence polynomial (and total number of independent sets) of cubic graphs of girth at least 5. The bound is achieved by unions of the Heawood graph, the point/line incidence graph of the Fano plane.

We also give a tight lower bound on the total number of independent sets of triangle-free cubic graphs. This bound is achieved by unions of the Petersen graph.

We conjecture that in fact all Moore graphs are extremal for the scaled number of independent sets in regular graphs of a given minimum girth, maximizing this quantity if their girth is even and minimizing if odd. The Heawood and Petersen graphs are instances of this conjecture, along with complete graphs, complete bipartite graphs, and cycles.

1. Independent sets in regular graphs

A classic theorem of Kahn [10] states that a union of $n/2d$ copies of the complete d-regular bipartite graph ($K_{d,d}$) has the most independent sets of all d-regular bipartite graphs on n vertices. Zhao [13] extended this to all d-regular graphs. A result of Galvin

E-mail addresses: g.perarnau@bham.ac.uk (G. Perarnau), math@willperkins.org (W. Perkins).

[1] WP supported in part by EPSRC grant EP/P009913/1.

https://doi.org/10.1016/j.jctb.2018.04.009
0095-8956/

and Tetali [8] for bipartite graphs combined with Zhao's result shows that maximality of $K_{d,d}$ holds at the level of the *independence polynomial*,

$$P_G(\lambda) = \sum_{I \in \mathcal{I}(G)} \lambda^{|I|}, \tag{1}$$

where $\mathcal{I}(G)$ is the set of all independent sets of G.

Theorem 1 *(Kahn, Galvin–Tetali, Zhao [10,8,13]). For every d-regular graph G and all $\lambda > 0$,*

$$\frac{1}{|V(G)|} \log P_G(\lambda) \leq \frac{1}{2d} \log P_{K_{d,d}}(\lambda). \tag{2}$$

The result on the number of independent sets in a regular graph is recovered by taking $\lambda = 1$ and noting that the independence polynomial is multiplicative over taking disjoint unions of graphs.

The function $P_G(\lambda)$ is also known as the partition function (the normalizing constant) of the *hard-core model* from statistical physics. The hard-core model is a probability distribution over the independent sets of a graph G, parametrized by a positive real number λ, the *fugacity*. The distribution is given by:

$$\Pr[I] = \frac{\lambda^{|I|}}{P_G(\lambda)}.$$

The derivative of $\frac{1}{|V(G)|} \log P_G(\lambda)$ has a nice probabilistic interpretation: it is the *occupancy fraction*, $\alpha_G(\lambda)$, the expected fraction of vertices of G in the random independent set drawn from the hard-core model:

$$\begin{aligned}
\alpha_G(\lambda) &:= \frac{1}{|V(G)|} \mathbb{E}|I| \\
&= \frac{\lambda P'_G(\lambda)}{|V(G)| \cdot P_G(\lambda)} \\
&= \frac{\lambda}{|V(G)|} (\log P_G(\lambda))'.
\end{aligned}$$

Davies, Jenssen, Perkins and Roberts recently gave a strengthening of Theorem 1, showing that (2) holds at the level of the occupancy fraction.

Theorem 2 *(Davies, Jenssen, Perkins, Roberts [5]). For every d-regular graph G and all $\lambda > 0$,*

$$\alpha_G(\lambda) \leq \alpha_{K_{d,d}}(\lambda). \tag{3}$$