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# On the inducibility of cycles ☆

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ABSTRACT

In 1975 Pippenger and Golumbic proved that any graph on $n$ vertices admits at most $2e(n/k)^k$ induced $k$-cycles. This bound is larger by a multiplicative factor of $2e$ than the simple lower bound obtained by a blow-up construction. Pippenger and Golumbic conjectured that the latter lower bound is essentially tight. In the present paper we establish a better upper bound of $(128e/81) \cdot (n/k)^k$. This constitutes the first progress towards proving the aforementioned conjecture since it was posed.



## 1. Introduction

A common theme in modern extremal combinatorics is the study of densities or induced densities of fixed objects (such as graphs, digraphs, hypergraphs, etc.) in large objects of the same type, possibly under certain restrictions. This general framework includes Turán densities of graphs and hypergraphs, local profiles of graphs and their relation to quasi-randomness, and more. One such line of research was initiated by Pippenger and Golumbic [16]. Given graphs $G$ and $H$, let $D_H(G)$ denote the number of induced subgraphs of $G$ that are isomorphic to $H$ and let

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$I_H(n) = \max\{D_H(G) : |G| = n\}$. A standard averaging argument was used in [16] to show that the sequence $\{I_H(n)/\binom{n}{|H|}\}_{n=|H|}^{\infty}$ is monotone decreasing, and thus converges to a limit $ind(H)$, the so-called *inducibility* of $H$.

Since it was first introduced in 1975, inducibility has been studied in many subsequent papers. Determining this invariant seems to be a very hard problem. To illustrate the current state of knowledge (or lack thereof), it is worthwhile to note that even the inducibility of paths of length at least 3 and cycles of length at least 6 are not known. Still, the inducibility of a handful of graphs and graph classes is known. These include various very small graphs (see, e.g., [1,7,13]) and complete multipartite graphs (see, e.g., [3–5]). Additional recent results on inducibility can be found, e.g., in [11,14,15]. Some of the recent progress in this area is due to Razborov's theory of flag algebras [17], which provides a framework for systematic computer-aided study of questions of this type.

While, trivially, the complete graph $H = K_k$ and its complement achieve the maximal possible inducibility of 1, the natural analogous question, which graphs on $k$ vertices *minimise* the quantity $ind(H)$, which has been asked in [16], is still open.

Let $H$ be an arbitrary graph on $k$ vertices, where $k$ is viewed as large but fixed. By considering a balanced blow-up of $H$ (and ignoring divisibility issues), it is easy to see that $ind(H) \geq k!/k^k$. An iterated blow-up construction provides only a marginally better lower bound of $k!/(k^k - k)$. Pippenger and Golumbic [16] conjectured that the latter is tight for cycles.

**Conjecture 1.1** *([16]).* $ind(C_k) = k!/(k^k - k)$ *for every* $k \geq 5$.

Note that the requirement $k \geq 5$ appearing in Conjecture 1.1 is necessary. Indeed, $ind(C_3) = 1$ since $C_3 = K_3$ is a complete graph and, as shown in [16], $ind(C_4) = 3/8$ since $C_4 = K_{2,2}$ is a balanced complete bipartite graph. The authors of [16] also posed the following asymptotic version of the above conjecture.

**Conjecture 1.2** *([16]).* $ind(C_k) = (1 + o(1))k!/k^k$.

In support of Conjecture 1.2, it was shown in [16] that $I_{C_k}(n) \leq \frac{2n}{k}\left(\frac{n-1}{k-1}\right)^{k-1}$ holds for every $k \geq 4$. This implies that $ind(C_k) \leq 2e \cdot k!/k^k$, leaving a multiplicative gap of $2e$ (which is approximately 5.4366) between the known upper and lower bounds. In this paper we partially bridge the above gap by proving a better upper bound on the inducibility of $C_k$, namely $ind(C_k) \leq (128/81)e \cdot k!/k^k$ (note that $(128/81)e$ is approximately 4.2955).

**Theorem 1.3.** *For every* $k \geq 6$ *we have*

$$ind(C_k) \leq \frac{128e}{81} \cdot \frac{k!}{k^k}.$$

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