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Density of multivariate homogeneous polynomials on star like domains

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ABSTRACT

The famous Weierstrass theorem asserts that every continuous function on a compact set in $\mathbb{R}^d$ can be uniformly approximated by algebraic polynomials. A related interesting problem consists in studying the same question for the important subclass of *homogeneous* polynomials containing only monomials of the same degree. The corresponding conjecture claims that every continuous function on the boundary of *convex* 0-symmetric bodies can be uniformly approximated by pairs of homogeneous polynomials. The main objective of the present paper is to review the recent progress on this conjecture and provide a new unified treatment of the same problem on *non convex star like domains*. It will be shown that the boundary of every 0-symmetric non convex star like domain contains an exceptional zero set so that a continuous function can be uniformly approximated on the boundary of the domain by a sum of two homogeneous polynomials if and only if the function vanishes on this zero set. Thus the Weierstrass type approximation problem for homogeneous polynomials on non convex star like domains amounts to the study of these exceptional zero sets. We will also present an extension of a theorem of Varjú which describes the exceptional zero sets for intersections of star like domains. These results combined with certain transformations of the underlying region will lead to the discovery of some new classes of convex and non convex domains for which the Weierstrass type approximation result holds for homogeneous polynomials.

1. Introduction

The basic question of approximation theory concerns the possibility of approximation. Is the given family of functions from which we plan to approximate dense in the set of functions we wish to approximate? The first significant density results were those of Weierstrass who proved in 1885 the density of algebraic polynomials in the class of continuous real-valued functions on a compact interval, and the density of trigonometric

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polynomials in the class of 2π-periodic continuous real-valued functions. These classical Weierstrass approximation theorems led to numerous generalizations which were applied to other families of functions. They gave rise to the development of general methods for determining density namely, the Stone–Weierstrass theorem generalizing the above Weierstrass theorem to subalgebras of $C(X), X$ a compact space. In particular, the Stone–Weierstrass theorem yields the multivariate version of the classical Weierstrass theorem asserting that for any compact set $K \subset \mathbb{R}^d$ and any continuous real valued function $f \in C(K)$ there is a sequence of polynomials $p_n \in P_n^d$ of degree at most n such that $\lim_{n\to\infty} p_n = f$ uniformly on K. Here and in what follows P_n^d denotes the set of algebraic polynomials of degree at most n in d real variables. For a comprehensive treatment of density results in Approximation theory see the nice survey by Pinkus [7]. Of course, the most interesting density problems correspond to those situations when the subalgebra property fails and thus the Stone–Weierstrass theorem is not applicable. For instance, consider the linear space $M := \text{span}\{x^{\lambda_j}, 0 = \lambda_0 < \lambda_1 < ... \uparrow \infty\}$. Then M is a linear subspace of $C[0,1]$ which is not a subalgebra, because it is not close relative to multiplication, so the Stone–Weierstrass theorem can not be used. By the famous Müntz theorem M is dense in $C[0,1]$ if and only if $\sum_j \frac{1}{\lambda_j} = \infty$. Another relevant example is the Lorentz type set of *incomplete* polynomials $p_n(x) = \sum_{n\theta \le k \le n} a_k x^k,\ n \in \mathbb{N}$ where $0 < \theta < 1$ is a fixed number. This time the set of all these incomplete polynomials is closed relative to multiplication, but clearly it is not linear, i.e., the subalgebra condition fails again. It was shown by G.G. Lorentz, von Golitschek, and Saff and Varga that given $f \in C[0,1]$ there exists a sequence of incomplete polynomials which converges to f uniformly on $[0,1]$ if and only if the function vanishes on $[0,\theta^2]$, see [6], pp. 86–88. Hence this time in order to compensate for the lack of the subalgebra property one needs to impose an additional restriction that the functions vanish on a certain set. As shown in [9] these exceptional zero sets are typical in general in case when approximating by algebraic polynomials with varying weights. These phenomena of exceptional zero sets will also play a central part in our study of approximation by homogeneous polynomials on 0-symmetric star like domains.

2. On density of homogeneous polynomials on convex domains

In this paper we will consider the interesting and difficult problem related to the density of multivariate **homogeneous** polynomials

$$H_n^d := \{\sum_{|\mathbf{k}|=n} a_{\mathbf{k}}\mathbf{x}^{\mathbf{k}} : a_{\mathbf{k}} \in \mathbb{R}\},\ \ \mathbf{x} \in \mathbb{R}^d,\ \ H^d := \cup_n H_n^d.$$

Homogeneous polynomials $h \in H_n^d$ of degree n contain only monomials of exact degree n, and therefore they evidently satisfy the property $h(t\mathbf{x}) = t^n h(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}^d$ and $t \in \mathbb{R}$. Hence if $h_n(\mathbf{x}), n \in \mathbb{N}$ converge to a nonzero value at some $\mathbf{x} \in \mathbb{R}^d$, then they tend to zero at $t\mathbf{x}$ if $|t| < 1$ and tend to infinity for $|t| > 1$. Thus we must assume that each line that goes through the origin intersects the underlying domain in at most two points. Since homogeneous polynomials are either even or odd depending on their degree it is natural to consider compact sets symmetric with respect to the origin. So in view of the above comments we will restrict our attention to *0-symmetric star like domains* $K \subset \mathbb{R}$ which satisfy the property that for every $\mathbf{x} \in K$ we have $(-\mathbf{x}, \mathbf{x}) \in \text{IntK}$ and will study the approximation problem on the boundary ∂K of this 0-symmetric star like domain K. In addition, we clearly need in general both even and odd polynomials in order to approximate arbitrary continuous functions, so the density problem will be considered for sums of pairs of homogeneous polynomials.

Example. Consider the unit sphere in $\mathbb{R}^d$ given by:

$$S^{d-1} = \{\mathbf{x} = (x_1, ..., x_d) \in \mathbb{R}^d : x_1^2 + ... + x_d^2 = 1\}.$$