J. Math. Anal. Appl. 469 (2019) 280–297

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Necessary conditions in stochastic linear quadratic problems and their applications ☆

Tianxiao Wang

School of Mathematics, Sichuan University, Chengdu, PR China

ARTICLE INFO

Article history:
Received 10 May 2018
Available online 11 September 2018
Submitted by H. Zwart

Keywords:
Stochastic linear quadratic problems
Closed-loop optimal strategies
Open-loop optimal controls
Variational approach

ABSTRACT

Stochastic linear quadratic optimal control problems are considered. A unified approach is proposed to treat the necessary optimality conditions of closed-loop optimal strategies and open-loop optimal controls. Notice that the former notion does not rely on initial wealth, while the later one does. Our conclusions of closed-loop optimal strategies are directly derived by suitable variational methods, the approach to which is different from [12], [11]. Moreover, the necessary conditions for closed-loop optimal strategies happen to be sufficient which takes us by surprise. Finally, two applications are given as illustration.

1. Introduction

Suppose $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ is a complete filtered probability space, $W(\cdot)$ is a one-dimensional standard Brownian motion with natural filtration $\mathbb{F} \equiv \{\mathcal{F}_t\}_{t\geq 0}$ augmented by all $\mathbb{P}$-null sets. We consider the following stochastic differential equation (SDE):

$$\begin{cases} dX(s) = \big[A(s)X(s) + B(s)u(s) + b(s)\big]ds + \big[C(s)X(s) + D(s)u(s) + \sigma(s)\big]dW(s), \\ X(0) = x, \end{cases} \tag{1.1}$$

with $s \in [0, T]$ and a quadratic cost functional

$$\begin{aligned} J(u(\cdot); 0, x) = \frac{1}{2}\mathbb{E}\Big\{ \int_0^T \big[\langle Q(s)X(s), X(s)\rangle + \langle R(s)u(s), u(s)\rangle\big]ds \\ + \langle GX(T), X(T)\rangle \Big\} + \langle \gamma_2, \mathbb{E}X(T)\rangle. \end{aligned} \tag{1.2}$$

☆ The research was supported by the NSF of China under grant 11231007, 11401404 and 11471231, and the Fundamental Research Funds for the central Universities (YJ201605).

E-mail address: wtxiao2014@scu.edu.cn.

https://doi.org/10.1016/j.jmaa.2018.09.013
0022-247X/

Here $A(\cdot), B(\cdot), C(\cdot), D(\cdot), Q(\cdot), R(\cdot)$ are suitable matrix-valued (deterministic) functions, $b(\cdot)$, $\sigma(\cdot)$ are stochastic processes, G is $\mathbb{R}^{n\times n}$-matrix, $\gamma_2 \in \mathbb{R}^n$. Under (H1), see Section 2, for any $x \in \mathbb{R}^n$, $u(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$, there exists a unique $X(\cdot) \in L^2_{\mathbb{F}}(\Omega; C([0,T];\mathbb{R}^n))$ satisfying (1.1). Consequently, (1.2) is well-defined and we state the stochastic linear quadratic (SLQ, in short) optimal control problem as follows.

Problem (SLQ). For any given initial state $x \in \mathbb{R}^n$, find $\bar{u}(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)$ such that

$$J(\bar{u}(\cdot); 0, x) = \inf_{u(\cdot)\in L^2_{\mathbb{F}}(0,T;\mathbb{R}^m)} J(u(\cdot); 0, x). \tag{1.3}$$

Optimal linear quadratic problem was firstly studied in [8]. In the deterministic case, the control problem can be solved elegantly via the Riccati equation if $R(\cdot)$ is uniformly positive definite (see [16]). As to stochastic case, it was firstly discussed in [15], and were later studied in several other papers. In those works, $R(s) > 0$ was taken for granted until the work of [2]. It is pointed out in [2] that **Problem (SLQ)** might still be solvable if $R(\cdot)$ is negative. Some following-up works include [3], [4], [6]. For SLQ problem with random coefficients, we further refer to [10], [5], [13]. On the other hand, Ait Rami et al. [1] introduced a generalized Riccati equations involving pseudo-inverse of a matrix and an additional algebraic constraint. More recently, [12] and [11] studied the optimal control problem from the view of closed-loop optimal strategies, and gave their characterization via the solvability of certain generalized Riccati equations.

In this paper, we also study the closed-loop optimal strategies of **Problem (SLQ)**. From this point of view, this paper can be regarded as a continuation of [12], [11] in a certain sense. It is well-known that the first-order, second-order necessary condition of optimal controls can be derived by stochastic maximum principles which essentially relies on the spike variation. Hence can we directly apply similar variational ideas to investigate the case of closed-loop optimal strategies? Our main aim of this manuscript is to establish the necessity conditions of closed-loop optimal strategies with a new variational manner. To do it, we use the ideas of dynamic programming principle to transform the *stationary* optimality in $[0,T]$ into a *dynamic* version in $[t,T]$ with $t \in [0,T]$. After that, we introduce a particular perturbation of closed-loop optimal outcome control process in $[t, t+\varepsilon]$. Notice that this optimal outcome control is a linear form of optimal state and is defined by the closed-loop optimal strategy. We also develop the corresponding duality techniques and decoupling arguments. It is worthy mentioning that this approach can be adopted to discuss some time inconsistent optimal control problems (e.g. [7], [14]).

Among other things, we find two interesting facts which are stated below. First of all, the obtained necessary conditions of closed-loop optimal strategies turn out to be sufficient as well. In other words, we use the variational arguments, which are widely used in obtaining maximum principles, to obtain a characterization of closed-loop optimal strategies. Second, our introduced variation can reduce to classical spike variation, which enables us to derive the analogue results for open-loop optimal controls. Consequently, we give a unified treatment of both open-loop optimal controls and closed-loop optimal strategies.

This paper is organized as follows. In Section 2, we provide some preliminary notations and conclusions. In Section 3, after some preparations we prove the main results. In Section 4, we indicate how these techniques and conclusions can be used to study other problems. Section 5 concludes the paper.

2. Preliminaries

Given (1.1) and (1.2), we introduce the following hypotheses.

(H1). Let $A(\cdot), C(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times n})$, $Q(\cdot) \in L^\infty(0,T;\mathbb{S}^{n\times n})$, $B(\cdot), D(\cdot) \in L^\infty(0,T;\mathbb{R}^{n\times m})$, $G \in \mathbb{S}^{n\times n}$, $\gamma_2 \in \mathbb{R}^n$, $b(\cdot) \in L^2_{\mathbb{F}}(\Omega; L^1(0,T;\mathbb{R}^n))$, $\sigma(\cdot) \in L^2_{\mathbb{F}}(0,T;\mathbb{R}^n)$, $R(\cdot) \in L^\infty(0,T;\mathbb{S}^{m\times m})$.

Here $\mathbb{S}^{m\times m}$ is the set of symmetric $m \times m$ matrices. For $0 \le s \le t \le T$, $H := \mathbb{R}^n, \mathbb{R}^{n\times n}$, etc, we define the following spaces.