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Trilinear L^p estimates with applications to the Cauchy problem for the Hartree-type equation

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ABSTRACT

In this paper, L^p estimates for a trilinear operator associated with the Hartree type nonlinearity are proved. Moreover, as application of these estimates, it is proved that after a linear transformation, the Cauchy problem for the Hartree-type equation becomes locally well posed in the Bessel potential and homogeneous Besov spaces under certain regularity assumptions on the initial data. This notion of well-posedness and the functional framework to solve the equation were firstly proposed by Y. Zhou.

1. Introduction

In this paper we discuss the Cauchy problem for the Hartree type equation

$$(H)\begin{cases} iu_t + \Delta u + (K * |u|^2)u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

where $K(x) = |x|^{-\gamma}$ and $0 < \gamma < n$, $n \in \mathbb{N}$, and the initial data u_0 are in the Bessel potential space H_p^s or homogeneous Besov space $\dot{B}^s_{p,1}$.

It is well known that the Cauchy problem for the cubic nonlinear Schrödinger equation

$$(C)\begin{cases} iu_t + \Delta u + |u|^2 u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases}$$

is locally well posed for data in H^s under certain conditions on s (see [4], [5], [9], [14], [16], [19]). In particular, the Cauchy problem is locally well posed in $H^0 = L^2$ when $n = 1, 2$. On the other hand, it is widely believed

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by many researchers that the Cauchy problem is ill posed in L^p-spaces when $p \neq 2$. This is probably because it was proved that the Schrödinger equation is ill posed in L^p, $p \neq 2$ even in the linear case. Indeed, to see this for $p < 2$, let $e^{it\Delta}\varphi$ be the solution of the linear Cauchy problem

$$iu_t + \Delta u = 0, \quad u(0) = \varphi \in L^p \setminus L^2,$$

and assume that $e^{it_0\Delta}\varphi \in L^p$ for some $t_0 \neq 0$. Then it is known that $e^{it_0\Delta}\varphi \in L^{p'}$, which implies $e^{it\Delta}\varphi \in L^2$. By the unitary property, this implies $\varphi \in L^2$, which is a contradiction.

Recently, however, there have been some attempts to solve (C) or other nonlinear dispersive equations for data in X, where X is not characterized by any kind of square integrability. For example, Grünrock [11] considered the space

$$\widehat{H^s_p} = \{\varphi \in \mathcal{S}'(\mathbb{R}^n) \ \ |(1 + |\xi|^2)^{\frac{s}{2}}\hat{\varphi} \in L^{p'}\}$$

as a class of initial data and proved that (C) is locally well posed in $\widehat{H^s_p}(\mathbb{R}^1)$ for $s \geq 0$, $1 < p < \infty$. The space $\widehat{H^s_p}$ scales like H^s_p and by the Hausdorff–Young inequality,

$$\begin{aligned} H^s_p &\subseteq \widehat{H^s_p}, \quad 1 \leq p \leq 2 \\ \widehat{H^s_p} &\subseteq H^s_p, \quad 2 \leq p \leq \infty, \end{aligned}$$

which implies $\widehat{H^s_p}$ spaces are similar to H^s_p and are good substitutes for H^s_p. Therefore, treating in some kind of square integrable framework is not necessarily required for the well-posedness property of the Cauchy problem for the nonlinear dispersive equations. Nevertheless, one faces great deal of difficulty when trying to obtain any kind of well-posedness results for the nonlinear dispersive equations in mere L^p-spaces for $p \neq 2$ due to the reason mentioned above. One way to overcome this difficulty is to introduce some, in a sense, "weaker" notion of well-posedness. For instance, in 2010 YI. Zhou [21] considered (C) with data in the Sobolev–Slobodekij space W^s_p (which is understood as the Besov space $B^s_{p,p}$, strictly speaking) and discovered that after a certain linear transformation the Cauchy problem becomes well posed in W^s_p. More precisely, considering the integral equation corresponding to (C)

$$\text{(I)} \quad u(t) = e^{it\Delta}u_0 + i\int_0^t e^{i(t-\tau)\Delta}(|u(\tau)|^2 u(\tau))d\tau$$

and introduce $v(t) \triangleq e^{-it\Delta}u(t)$ and rewrite (I) in terms of v. Then the integral equation is equivalent to:

$$\text{(I)}^* \quad v(t) = u_0 + i\int_0^t e^{-i\tau\Delta}\big((e^{i\tau\Delta}v(\tau))(\overline{e^{i\tau\Delta}v(\tau)})(e^{i\tau\Delta}v(\tau))\big)d\tau.$$

In his work [21], Y. Zhou showed that the integral equation is locally well posed in W^s_p under certain conditions on p, s, n, including the case of $n = 1, s = 0$, that is, $L^p(\mathbb{R})$. His work relies on a "good" cancellation property of the trilinear form

$$T(u(t), v(t), w(t)) = e^{-it\Delta}((e^{it\Delta}u(t))(e^{it\Delta}v(t))(\overline{e^{it\Delta}w(t)})). \tag{1.1}$$

Indeed, exploiting this property, one can prove the trilinear L^1 estimate

$$\|T(u(t), v(t), w(t))\|_{L^1} \leq C|t|^{-n}\|u(t)\|_{L^1}\|v(t)\|_{L^1}\|w(t)\|_{L^1}. \tag{1.2}$$