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Vector-valued nonstationary Gabor frames ☆

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A B S T R A C T

As an extension of Gabor frames, nonstationary Gabor (NSG) frames were recently introduced in adaptive signal analysis. They allow for efficient reconstruction with flexible sampling and varying window functions. In this paper we generalize the notion of NSG frames from $L^2(\mathbb{R})$ to the vector-valued Hilbert space $L^2(\mathbb{R}, \mathbb{C}^L)$, and investigate the resulting vector-valued NSG frames. We derive a Walnut's representation of the mixed frame operator, and provide some necessary/sufficient conditions for a vector-valued NSG system to be a frame for $L^2(\mathbb{R}, \mathbb{C}^L)$. Furthermore, we show the existence of painless vector-valued NSG frames, and of vector-valued NSG frames with fast decaying window functions.

1. Introduction

Let $g \in L^2(\mathbb{R})$ and $a, b > 0$. The corresponding Gabor system is the set of functions $\{M_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ with the translation and modulation operators given by $T_\mu g(\cdot) = g(\cdot - \mu)$ and $M_\lambda g(\cdot) = e^{2\pi i\lambda\cdot}g(\cdot)$ respectively. The function g is called the *window function* or the *generator*, and a, b are called the *time-frequency shift parameters*. Of particular interest are Gabor systems that allow for stable, perfect reconstruction of any functions $f \in L^2(\mathbb{R})$ from the coefficients $\langle f, M_{mb}T_{na}g\rangle$, $m, n \in \mathbb{Z}$. Such Gabor systems are called Gabor frames. They have dual frames with the same structure. This remarkable property does not hold for more general frames and is one important reason for the elegance of Gabor frames in both theory and applications [11], [16], [17], [20].

Although Gabor frames have found a wide range of applications, they provide a fixed time-frequency resolution and sampling strategy, which is often undesirable in processing signals with variable time-frequency characteristics. To overcome this deficit, various adapted and adaptive signal representations have been introduced (see [7] and references therein). These representations allow for varying window functions or sampling strategy over time, frequency or both. Adaption over frequency leads, for example, to non-uniform

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filter banks [38], while adaption over time is considered in approaches such as modulated lapped transforms [32], [35], adapted local trigonometric transforms [41] and time-varying wavelet packets [36]. Transforms featuring simultaneous adaptivity in time and frequency have been shown to be useful in some applications [30], [31], but they are notoriously difficult to construct and implement [12], [26], [37].

Most of the cited work achieves flexible tilings of the time-frequency plane, however methods that allow for efficient reconstruction with flexible sampling and varying window functions are scarce. One construction that unites these desirable properties is the concept of *nonstationary Gabor* (NSG) systems, first proposed by Jaillet in [25]. NSG systems generalize Gabor systems of regular time-frequency shifts of a single window function by allowing for arbitrary window functions to cover the time-axis and, correspondingly, frequency shift parameters that change in time. Explicitly, let $\{g_n\}_{n\in\mathbb{Z}}$ be a set of window functions in $L^2(\mathbb{R})$ and let $\{b_n\}_{n\in\mathbb{Z}}$ be a corresponding sequence of frequency shift parameters, then the set $\{M_{mb_n}g_n\}_{m,\,n\in\mathbb{Z}}$ is called an NSG system. In particular, an NSG system with $g_n = T_{na}g$ and $b_n = b$ for all $n \in \mathbb{Z}$ with $g \in L^2(\mathbb{R})$ and $a, b > 0$ is a Gabor system. It should be noticed that, NSG systems allow for the usage of different window functions and irregular sampling points, as opposed to merely irregular sampling points, which is the situation most frequently addressed in the literature (see [9], [10] [18], [39] and references therein). Meanwhile, the regular frequency sampling points at each temporal position guarantee the fast, FFT-based implementation.

If an NSG system constitutes a frame, we call it an NSG frame. NSG frames combine the adaptivity of local Fourier bases [3], [32] with the flexibility of redundant systems to provide a powerful framework for time-frequency representations. Painless NSG frames were constructed in [6] and used in realizing various time- or frequency-adaptive transforms [6], [24], [31], [33], [34], [40]. The theory of NSG frames beyond the painless case was developed in [13] and [14], where NSG frames with fast decaying window functions were considered and approximately dual frames were derived from sufficiently close known NSG frames. Note that the existence of a dual frame with the same structure is not guaranteed for general NSG frames. Holighaus in [23] investigated the structural properties of dual systems for NSG frames, and showed that whenever an NSG system $\{M_{mb_n}g_n\}_{m,\,n\in\mathbb{Z}}$, comprised of compactly supported window functions with moderate overlap and sufficiently small frequency shift parameters, constitutes a frame, the canonical dual frame $\{\widetilde{g_{m,\,n}}\}_{m,\,n\in\mathbb{Z}}$ is not too different in structure from $\{M_{mb_n}g_n\}_{m,\,n\in\mathbb{Z}}$ itself, that is, $\widetilde{g_{m,\,n}}$ is compactly supported for $m, n \in \mathbb{Z}$, and

$$\widetilde{g_{m,\,n}} = M_{mb_n}\Bigg(\widetilde{g_{0,\,n}}\,\Big|_{[c_n,\,d_n]} + \sum_{k\in\mathbb{N}} M_{mb_n\sum_{j=0}^{k-1}b_{n-k+j}^{-1}}\widetilde{g_{0,\,n}}\,\Big|_{\left[c_{n-k},\,d_{n-1}-\sum_{j=0}^{k-1}b_{n-k+j}^{-1}\right]} + \sum_{k\in\mathbb{N}} M_{-mb_n\sum_{j=0}^{k-1}b_{n+k-j}^{-1}}\widetilde{g_{0,\,n}}\,\Big|_{\left[c_{n+1}+\sum_{j=0}^{k-1}b_{n+k-j}^{-1},\,d_{n+k}\right]}\Bigg)$$

for $m \in \mathbb{Z}\backslash\{0\}$ and $n \in \mathbb{Z}$, where $\mathrm{supp}(g_n) = [c_n, d_n]$.

Let L be a given positive integer. In this paper, we generalize the notion of NSG systems from $L^2(\mathbb{R})$ to the space $L^2(\mathbb{R}, \mathbb{C}^L)$ of vector-valued signals. Elements of $L^2(\mathbb{R}, \mathbb{C}^L)$ can also be understood as vectors $\mathbf{f} = (f_1, f_2, \ldots, f_L)$ with $f_\ell \in L^2(\mathbb{R})$, which amounts to identifying $L^2(\mathbb{R}, \mathbb{C}^L)$ with the L-fold direct sum of $L^2(\mathbb{R})$. The inner product is given by

$$\langle \mathbf{f}, \tilde{\mathbf{f}}\rangle = \sum_{\ell=1}^{L}\langle f_\ell, \tilde{f}_\ell\rangle_{L^2(\mathbb{R})} = \sum_{\ell=1}^{L}\int_{\mathbb{R}} f_\ell(x)\overline{\tilde{f}_\ell(x)}dx$$

for $\mathbf{f} = (f_1, f_2, \ldots, f_L)$, $\tilde{\mathbf{f}} = (\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_L) \in L^2(\mathbb{R}, \mathbb{C}^L)$. In what follows, for $\mathbf{f} \in L^2(\mathbb{R}, \mathbb{C}^L)$ and $1 \le \ell \le L$, we always denote by f_ℓ the ℓ-th component of $\mathbf{f}$. Let

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