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# Convergence of Cauchy sequences for the covariant Gromov–Hausdorff propinquity

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A B S T R A C T

The covariant Gromov–Hausdorff propinquity is a distance on Lipschitz dynamical systems over quantum compact metric spaces, up to equivariant full quantum isometry. It is built from the dual Gromov–Hausdorff propinquity which, as its classical counterpart, is complete. We prove in this paper several sufficient conditions for convergence of Cauchy sequences for the covariant propinquity and apply it to show that many natural classes of dynamical systems are complete for this metric.


## 1. Introduction

The covariant Gromov–Hausdorff propinquity is a distance, up to equivariant full quantum isometry, on the class of Lipschitz dynamical systems, defined as the class of quantum compact metric space endowed with a strongly continuous action of a proper monoid by Lipschitz morphisms or even Lipschitz linear maps. The class of Lipschitz dynamical systems include C*-dynamical systems by Lipschitz automorphisms as well as actions by monoids of completely positive maps which map the domain of the L-seminorm of a quantum compact metric space to itself. We proved in [19] that the covariant propinquity is a metric up to equivariant full quantum isometry — namely, distance zero implies the existence of a full quantum isometry between the quantum compact metric spaces as well as an isometric isomorphism between the acting monoids, which intertwine the actions in a natural fashion. We illustrate in [19] our metric by showing that fuzzy tori with their dual actions converge to quantum tori with their own dual actions for the covariant propinquity.

The covariant propinquity is built from the dual Gromov–Hausdorff propinquity [15,11,9,18,16,12], which actually enjoys some natural covariance properties [17], though it is only defined on quantum compact metric spaces and thus does not fully capture the structure of a Lipschitz dynamical system. Nonetheless, our work

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in [17] suggests that a covariant propinquity, as introduced in [19], is a natural object to construct. The covariant propinquity between two Lipschitz dynamical systems dominate the propinquity between the underlying quantum compact metric spaces and the pointed Gromov–Hausdorff distance [5] between the underlying proper monoids, and both these last two distances are in particular complete. We are thus left with a very natural question: what classes of Lipschitz dynamical systems are complete when endowed with the covariant propinquity?

This question is the subject of the present paper. As the covariant propinquity is built using a covariant version of the pointed Gromov–Hausdorff distance between proper monoids, we begin with finding natural classes of proper monoids complete for the monoid-adapted Gromov–Hausdorff distance. We then discover that completeness is not a trivial matter, and in fact, we require a form of equicontinuity of the right translations of our monoids to provide a sufficient condition on Cauchy sequences to converge. We also see that additional complications arise when working with proper groups. We are however able to establish a generous sufficient condition which applies to a large class of natural examples, especially arising from actions on quantum compact metric spaces.

We then turn to the matter of convergence for Cauchy sequences for the covariant propinquity. We provide a sufficient condition which includes the condition exhibited for the monoid-Gromov–Hausdorff distance, and a similar condition on the actions themselves. The reason for this double condition is simply that we actually allow for a metric on proper monoids of Lipschitz dynamical systems which may not be the same as the natural pseudo-metric induced by the quantum compact metric space structure of the space on which the monoid acts. We explain this matter toward the end of this paper. We are in fact able to prove a stronger result than completeness for some classes of Lipschitz dynamical systems: we exhibit a sufficient condition for sequential compactness of certain classes of Lipschitz dynamical systems based upon the inherent covariant properties of the dual propinquity. From this result, our completeness result derives. We do know of a direct proof of our completeness result which does not involve the compactness properties we prove in this paper, but our proof of completeness does not weaken the assumptions we make here, and thus this approach is the most potent we know at this moment.

We begin our paper with a background section on noncommutative metric geometry and the covariant propinquity to set up the framework of this paper.

## 2. The covariant Gromov–Hausdorff propinquity

The covariant propinquity is defined on Lipschitz dynamical systems, which are proper monoid actions on quantum compact metric spaces by Lipschitz maps. A quantum compact metric space is a noncommutative analogue of the algebra of Lipschitz functions over a compact metric space [4,22,23,7,8,15].

**Notation 2.1.** Throughout this paper, for any unital C*-algebra $\mathfrak{A}$, the norm of $\mathfrak{A}$ is denoted by $\|\cdot\|_{\mathfrak{A}}$, the space of self-adjoint elements in $\mathfrak{A}$ is denoted by $\mathfrak{sa}(\mathfrak{A})$, the unit of $\mathfrak{A}$ is denoted by $1_{\mathfrak{A}}$ and the state space of $\mathfrak{A}$ is denoted by $\mathscr{S}(\mathfrak{A})$. We also adopt the convention that if a seminorm $\mathsf{L}$ is defined on some dense subspace of $\mathfrak{sa}(\mathfrak{A})$ and $a \in \mathfrak{sa}(\mathfrak{A})$ is not in the domain of $\mathsf{L}$, then $\mathsf{L}(a) = \infty$.

**Definition 2.2.** A *quantum compact metric space* $(\mathfrak{A}, \mathsf{L})$ is an ordered pair of a unital C*-algebra $\mathfrak{A}$ and a seminorm $\mathsf{L}$, called an *L-seminorm*, defined on a dense Jordan–Lie subalgebra $\mathrm{dom}(\mathsf{L})$ of $\mathfrak{sa}(\mathfrak{A})$, such that:

1. $\{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) = 0\} = \mathbb{R}1_{\mathfrak{A}}$,
2. the *Monge–Kantorovich metric* $\mathsf{mk}_{\mathsf{L}}$ defined for any two states $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$ by:

$$\mathsf{mk}_{\mathsf{L}}(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : a \in \mathrm{dom}(\mathsf{L}), \mathsf{L}(a) \leqslant 1\}$$

metrizes the weak* topology restricted to $\mathscr{S}(\mathfrak{A})$,