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Global synchronization of partially forced Kuramoto oscillators on networks

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HIGHLIGHTS

- We study the partially forced Kuramoto model on networks.
- The force for global synchronization scales with the inverse of the forced fraction.
- Topology of the network and the set of forced nodes influences global synchronization.
- Synchronization with a given fraction requires a minimum internal coupling strength.
- We present numerical and analytical calculations.

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ABSTRACT

We study the synchronization of Kuramoto oscillators on networks where only a fraction of them is subjected to a periodic external force. When all oscillators receive the external drive the system always synchronize with the periodic force if its intensity is sufficiently large. Our goal is to understand the conditions for global synchronization as a function of the fraction of nodes being forced and how these conditions depend on network topology, strength of internal couplings and intensity of external forcing. Numerical simulations show that the force required to synchronize the network with the external drive increases as the inverse of the fraction of forced nodes. However, for a given coupling strength, synchronization does not occur below a critical fraction, no matter how large is the force. Network topology and properties of the forced nodes also affect the critical force for synchronization. We develop analytical calculations for the critical force for synchronization as a function of the fraction of forced oscillators and for the critical fraction as a function of coupling strength. We also describe the transition from synchronization with the external drive to spontaneous synchronization.

1. Introduction

Coupled biological oscillators are abundant in nature and often need to work in synchrony to regulate physical activities, such as pacemaker cells in the heart [1], neurons in regions of the brain [2–4] and fireflies flashing collectively to help females find suitable mates [5,6]. Artificial systems, such as electrochemical oscillators [7] and coupled metronomes [8], have also been studied. There are evidences that synchronization also plays a key role in information processing in areas on the cerebral cortex [9,10]. Even the brain rest state activity is characterized by local rhythmic synchrony that induces spatiotemporally organized spontaneous activity at the level of the entire brain [11].

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The model of coupled oscillators introduced by Kuramoto [12] has become a paradigm in the study of synchronization and has been extensively explored in the last years in connection with biological systems, neural networks and the social sciences [13,14]. The model consists of N oscillators described by internal phases θ_i which rotate with natural frequencies ω_i typically selected from a symmetric distribution. In the original model all oscillators interact with each other according to the equations

$$\dot{\theta}_i = \omega_i + \frac{\lambda}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i), \tag{1}$$

where λ is the coupling strength and $i = 1, \ldots, N$.

Kuramoto analyzed the system in the limit where N goes to infinity and showed that for small values of the coupling parameter the oscillators continue to move as if they were independent. However, as the coupling increases beyond a critical value, a finite fraction of oscillators start to move together, a behavior termed spontaneous synchronization. This fraction increases smoothly with the coupling, characterizing a second order phase transition in the limit of infinite oscillators. For large enough coupling the whole system oscillates on the same frequency, as if it were a single element.

Synchronization in many biological systems, however, is not spontaneous, but frequently depends on external stimuli. Information processing in the brain, for example, might be triggered by visual, auditory or olfactory inputs [2]. Different patterns of synchronized neuronal firing are observed in the mammalian visual cortex when subjected to stimuli [3]. In the sensomotor cortex synchronized oscillations appear with amplitude and spatial patterns that depend on the task being performed [3,4]. Synchronization of brain regions that are not directly related to the task in question can be associated to disorders like epilepsy, autism, schizophrenia and Alzheimer [15,16]. In the heart, cardiac synchronization is induced by specialized cells in the sinoatrial node or by an artificial pacemaker that controls the rhythmic contractions of the whole heart [17]. The periodic electrical impulses generated by pacemakers can be seen as an external periodic force that synchronizes the heart cells. Another example of driven system is the daily light-dark cycle on the organisms [18]. In mammalians, cells specialized on the sleep control exhibit intrinsic oscillatory behavior whose connectivity is still unknown [19]. The change in the light-dark cycle leads to a response in the circadian cycle mediated by these cells, which synchronize via external stimulus.

A natural extension of the Kuramoto model, therefore, is to include the influence of an external periodic force acting on the system [20–23]. In this work we consider systems where the oscillators' interconnections form a network and where the force acts only on a fraction of the oscillators. We are interested in the conditions for global synchronization as a function of the fraction of nodes being forced and how it depends on network topology. We show that the minimum force F_{crit} needed for global synchronization scales as $1/f$, where f is the fraction of forced oscillators, and it is independent of the internal coupling strength λ. However, in order to reach synchronization with fraction f a minimum internal strength is needed. The degree distribution of the network and the set of forced nodes modify the $1/f$ behavior in heterogeneous networks. We develop analytical approximations for F_{crit} as a function of the fraction f of forced oscillators and for the minimum fraction f_{crit} for which synchronization occurs as a function of λ. This paper is organized as follows: in Section 2 we describe the partially forced Kuramoto model and present the results of numerical simulations in Section 3. In Section 4 we discuss the analytical calculations for $F_{crit}(f)$ and $f_{crit}(\lambda)$ that take into account network topology and explain most of the simulations. We summarize our conclusions in Section 5.

2. The forced Kuramoto model on networks

Here we consider three modifications of the original Kuramoto model: first, to include the possibility that each oscillator interacts only with a subset of the other oscillators, the system will be placed on a network whose topology defines the interactions [24]; second, we include the action of an external periodic force [20–22] and; third, we allow the external force to act only on a subset of the oscillators, representing the 'interface' of the system that interacts with the 'outside' world, like the photo-receptor cells in the eye [3].

The system is described by the equations

$$\dot{\theta}_i = \omega_i + F\,\delta_{i,C}\sin(\sigma t - \theta_i) + \frac{\lambda}{k_i}\sum_{j=1}^{N} A_{ij}\sin(\theta_j - \theta_i), \tag{2}$$

where A_{ij} is the adjacency matrix defined by $A_{ij} = 1$ if oscillators i and j interact and zero if they do not; k_i is the degree of node i, namely $k_i = \sum_j A_{ij}$; F and σ are respectively the amplitude and frequency of the external force; and C is the subgroup of oscillators subjected to the external force. We have also defined $\delta_{i,C} = 1$ if $i \in C$ and zero otherwise and we shall call N_C the number of nodes in the set C.

Following [22] we get rid of the explicit time dependence by performing a change of coordinates to analyse the dynamics in a referential frame corotating with the driving force:

$$\phi_i = \theta_i - \sigma t \tag{3}$$

which leads to

$$\dot{\phi}_i = \omega_i - \sigma - F\,\delta_{i,C}\sin\phi_i + \frac{\lambda}{k_i}\sum_{j=1}^{N} A_{ij}\sin(\phi_j - \phi_i), \tag{4}$$