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Reasonable method to extract Fisher information from experimental data

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HIGHLIGHTS

- An extensive investigation on FI extraction from Hellinger Distance and Kullback–Leibler entropy is presented.
- Two constraints on quadratic fitting for FI in a dichotomic measurement model are provided.
- Higher order fitting for reasonable FI is suggested and demonstrated to be useful with recent optical experimental data.

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ABSTRACT

Fisher information (FI) plays a crucial role in quantum precision measurement and entanglement detection. Recently two methods have been suggested to extract it from experimental data: Hellinger distance and Kullback–Leibler entropy. In this paper, an extensive investigation is considered with the help of a dichotomic measurement model. It is found that the general quadratic fitting for FI with both methods has two constraints: one is the critical visibility $V_0 = \sqrt{2/3}$, the other is the smallest $\delta\theta$ for the dichotomic measurement. To relax them, we propose the higher-order fitting (fourth order considered), by which a reasonable FI is obtained from recent optical experimental data.

1. Introduction

The degree of distinguishability between two neighboring quantum states is suggested to be characterized by statistical distance in terms of a Riemannian metric [1]. In quantum parameter estimation theory, increasing statistical distance corresponds to more reliable distinguishability, and thus to the possibility of estimating weakest signals. Therefore, the statistical distance is used to explore the sensitivity with which a parameter can be estimated [2,3]. Statistically, the relationship between the sensitivity and statistical distance is revealed by the Cramér–Rao lower bound [4], where the Fisher information (FI) is simply interpreted as the square of "statistical speed" (the rate of the change of the absolute statistical distance among two nearby states) [5]. Replacing FI with its quantum version obtained by optimizing over all possible quantum measurements, quantum Cramér–Rao lower bound [6,7] is obtained, i.e., $\Delta\theta \geq 1/\sqrt{F_Q[\hat{\rho}_{\rm in}, \hat{J}_{\vec{n}}]}$, where $\Delta\theta$ is the standard deviation on any arbitrary estimator of the parameter θ, $\hat{\rho}_{\rm in}$ is the initial state or input state and $\hat{J}_{\vec{n}}$ is the Hermitian generator of the phase shift. As separable and entangled states have different statistical responses to the parameter, Quantum Fisher information (QFI) has been shown to witness entanglement [5,8], and further to quantify the quantum resource for

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high-precision metrology [9]. Besides, QFI is also applied in modern quantum technologies [10], including quantum phase transitions [11–16], quantum Zeno dynamics [17], and quantum information protocols [18], etc.

That is the reason why how to extract FI from a quantum system, especially from bare experimental data [19], recently became a hot task. Starting from the experimental probability distributions of some observable, the FI was firstly accessed from the curvature of Hellinger Distance (HD) [20], requiring a quadratic fitting on the data with respect to a small variation of θ. Very recently, it is also recommended to extract FI by evaluating Kullback–Leibler (KL) entropy [19]. As shown in Refs. [19–21], there are a few advantages for both methods, such as independence on the specific shape of the probability distributions and no limitation on particle numbers.

In the view of entanglement witness, the values of FI are actually expected to be less than or equal to the intrinsic value to avoid a wrong detection. Nevertheless, due to the restriction of experimental techniques, it is not easy to adjust parameter so small to meet the criteria for quadratic fitting (see Eqs. (6) and (7)). Therefore, for a relatively large $\delta\theta$, the reasonable FI is hard to be extracted by quadratic fitting, where higher orders interfere the process severely. In this paper, we mainly investigate how to efficiently extract FI under available experimental conditions. With a dichotomic measurement model and recent methods (HD and KL entropy), two constraints for quadratic fitting are presented. One is on the critical visibility and the other is on the smallest $\delta\theta$, i.e.,Eqs. (6) and (7). To relax them, the higher-order fitting (fourth order considered) is discussed and demonstrated to be useful with latest optical experimental data.

Our paper is structured as follows. In Section 2, we present two constraints on quadratic fitting for FI in a dichotomic measurement model. In Section 3, we propose the higher-order fitting to overcome the constraints. An application of two fitting methods with the optical experimental data [22] is presented in Section 4. Finally, we present our conclusions in Section 5.

2. Constraints on quadratic fitting for FI in a dichotomic measurement model

In a general unbiased parameter estimation, the FI is defined as [8],

$$F(\theta) \equiv \sum_{\varepsilon} p(\varepsilon|\theta)\left(\frac{\partial \log p(\varepsilon|\theta)}{\partial\theta}\right)^2, \tag{1}$$

where $p(\varepsilon|\theta) = \mathrm{Tr}[\hat{\rho}_\theta \hat{M}_\varepsilon]$ denotes the conditional probability depending on the final states $\hat{\rho}_\theta$ and positive-operator valued measure (POVM) $\hat{M}_\varepsilon$. Usually, it is not easy to have an analytical expression for $p(\varepsilon|\theta)$, especially in some cases [23] we cannot obtain it any more. However, from the experimental point of view, it is not hard to have the information about $p(\varepsilon|\theta)$ by repeated measurements with adjusting parameter θ. As pointed in [19], with this and the direct connection between FI and HD (or KL entropy) in the limit of $\delta\theta \to 0$, FI was successfully obtained by quadratic fitting [20,21]. More clearly, for given two neighboring probability distributions $p(\varepsilon|\theta)$ and $p(\varepsilon|\theta+\delta\theta)$, HD is defined as

$$D_{\mathrm{HD}} \equiv 1 - \sum_{\varepsilon}\sqrt{p(\varepsilon|\theta)p\left(\varepsilon|\theta+\delta\theta\right)}, \tag{2}$$

and the corresponding Kullback–Leibler entropy [24] (also called relative entropy [25]), a well-established information-theoretic measure for the discrepancy of two probability distributions, is given by

$$D_{\mathrm{KL}} \equiv \sum_{\varepsilon} p(\varepsilon|\theta)\ln\frac{p(\varepsilon|\theta)}{p(\varepsilon|\theta+\delta\theta)}. \tag{3}$$

It is easy to check that the coefficients of the second order $(\delta\theta)^2$ in series expansion of Eqs. (2) and (3) around any given θ contain Eq. (1) but with different ratio factors [26,27],

$$D_{\mathrm{HD}} = C_2^{\mathrm{HD}}\left(\frac{1}{8}F(\theta)\right)\delta\theta^2 + C_3^{\mathrm{HD}}\delta\theta^3 + C_4^{\mathrm{HD}}\delta\theta^4 + O(\delta\theta^5), \tag{4}$$

$$D_{\mathrm{KL}} = C_2^{\mathrm{KL}}\left(\frac{1}{2}F(\theta)\right)\delta\theta^2 + C_3^{\mathrm{KL}}\delta\theta^3 + C_4^{\mathrm{KL}}\delta\theta^4 + O(\delta\theta^5). \tag{5}$$

As such, it is suggested to extract FI by quadratic fitting: expanding D_{HD} (D_{KL}) to the second order. Obviously, the smaller the $\delta\theta$, the better the evaluation for FI. However, once $\delta\theta$ is not so small, it may induce a false value, larger or smaller than the true FI [28]. Therefore, the study of the criteria for quadratic fitting in extracting FI is necessary.

To explore this we consider a dichotomic measurement model, parity measurement on the multi-ion GHZ state [29], where the conditional probability can be expressed as $p(\pm|\theta) = (1 \pm V\cos(N\theta))/2$ with the total ion number N and the visibility V from the oscillating fringe acquired by measurement $\varepsilon_i \in \{+,-\}$ on the state. Here, "$+$ $(-)$" denotes the state with an even (odd) number of excitations. With this conditional probability the optimal FI can be obtained by maximizing $F(\theta) = V^2N^2\sin^2(N\theta)/\left(1 - V^2\cos^2(N\theta)\right)$ over all possible θ, i.e., $F_{\mathrm{opt}} = V^2N^2$ at $\theta_0 = \pi/(2N) + n\pi/N$, where $n = 0, 1, 2, \ldots$.

In Fig. 1, we plot the D_{HD} as black curves, D_{KL} as red curves, as well as the fitting curves for quadratic fitting (blue curves) and higher-order fitting (dashed lines) at θ_0 with two different visibility $V = 0.787$ (Fig. 1(a, b)) and $V = 0.950$ (Fig. 1(c, d)). As mentioned before, for small enough $\delta\theta$ we find no difference in extracting FI by comparing all of the curves. However, a