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Research paper

Integrability conditions and solitonic solutions of the nonlinear Schrödinger equation with generalized dual-power nonlinearities, PT-symmetric potentials, and space- and time-dependent coefficients

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ABSTRACT

We consider a generalized nonlinear Schrödinger equation with dual power law nonlinearities, complex potential, and position- and time-dependent strengths of dispersion and nonlinearities. Using a standard similarity transformation, we obtain the integrability conditions and solitonic solutions of this equation by mapping it to its homogeneous version. Using a modified similarity transformation, where a solution of the homogeneous equation, which we denote as a seed, enters also in the transformation operator, a wider range of exact solutions is obtained including cases with complex potentials. We apply these two transformations to obtain two exact solitonic solutions of the homogeneous nonlinear Schrödinger equation, which are derived here for the first time for a general power of the nonlinearities, namely the flat-top soliton and tanh solution. We discuss and derive explicit solutions to the experimentally relevant cases associated with parabolic and PT-symmetric potentials.

1. Introduction

The nonlinear Schrödinger equation (NLSE) with cubic and quintic nonlinearities is known to describe optical solitons in fibers and nonlinear photonic crystals [1] and matter-wave solitons in Bose-Einstein condensates [2]. The quintic nonlinearity is one among many other higher order effects that start to be important for very thin (femto-second) optical solitons and when three-body collisions in matter-wave solitons becomes non-negligable [1,2]. The evolution of optical solitons in dispersion-managed fibers [1] and matter-wave solitons in trapped Bose-Einstein condensates [2] is described by a NLSE with a parabolic potential. Furthermore, the evolution of optical solitons in materials with complex refractive index [3–5] and matter-wave solitons in the presence of dissipation, introduces complex potentials to the NLSE. The so-called PT-symmetric potential is a particularly interesting example of such potentials. PT-symmetric potentials possess parity and time reversal symmetry. It has been shown by Bender [6] that such potentials, though complex, have real eigenvalues. A

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requirement for a 1D complex potential to be PT-symmetric is that its real part is an even function and its imaginary part is an odd function of the spacial variable.

It is thus experimentally-relevant to consider a NLSE with power law nonlinearities and complex potentials. Many references have studied the NLSE with cubic and quintic nonlinearities [7–13], and with parabolic [14,15] and PT-symmetric [16–24] potentials using several analytical methods for nonlinear differential equations [25–30]. We consider here a more general NLSE than what has been considered so far. The generalization is in the power of the nonlinearities not being restricted to cubic and quintic, and in introducing position- and time-dependent strengths of the dispersion and nonlinearities, and in the presence of a general complex potential (See Eq. 1 below.). A standard similarity transformation is used to transform this equation to its homogeneous version without a potential and with constant strengths of dispersion and nonlinearities. Then we use traditional methods to solve the homogeneous NLSE where we obtain a flat-top and tanh solitons. Applying the similarity transformation on these two solutions, we obtain the analogue solutions for the inhomogeneous NLSE. Further progress is made here by introducing a slightly modified similarity transformation where the solution of the homogeneous NLSE, denoted as the seed solution, is also used to construct the transformation operator, similarly to the Darboux transformation [31]. This results in solution classes composed of infinite hierarchies of exact solutions to the inhomogeneous NLSE. One class is obtained by applying the similarity transformation on all solutions of the NLSE belonging to one class. The other class is obtained by repeated action of the similarity transformation. With this modification, it became possible to find solutions for cases that were not integrable using the standard similarity transformation. One such case is the NLSE with constant coefficients and general position-dependent potential. Using the standard similarity transformation, integrability restricts the real part of the potential to a quadratic potential and the imaginary part of the potential to a constant. In the modified transformation, the real part of the potential is arbitrary and the imaginary part of the potential is related to it ensuring that when the real part is even, the imaginary part is odd, for a wide range of potentials, which allow for integrable NLSEs with PT-symmetric potential.

The rest of the paper is organized as follows. In Section 2, we derive the operator that transforms the inhomogeneous NLSE to homogeneous one. This will also result in integrability conditions that relate the coefficients of the inhomogeneous NLSE. In Section 2.1, we focus on the special case of time-dependent coefficients where we show that, for such a case, the real part of the potential is restricted to a quadratic potential with vanishing imaginary part. In Section 2.2 , we derive two exact solitonic solutions of the homogeneous NLSE. In Section 2.3, we apply the similarity transformation on these two solutions. In Section 3, we present an alternative similarity transformation and show how it leads to a hierarchy of exact solutions in the presence of PT-symmetric potentials. We end in Section 4 with conclusions and outlook.

2. Similarity transformation

The dimensionless form of the nonlinear Schrödinger's equation (NLSE) with space- and time-dependent coefficients in dual-power law medium and external complex potential is given by

$$\begin{aligned} &iq_t(x,t) + f(x,t)\,q_{xx}(x,t) + \left(g_1(x,t)|q(x,t)|^{2n} + g_2(x,t)|q(x,t)|^{4n}\right)q(x,t) \\ &\quad = -(V(x,t) + iW(x,t))q(x,t), \end{aligned} \tag{1}$$

where $q(x, t)$ is a complex valued wavefunction, $V(x, t)$ and $W(x, t)$ are arbitrary real functions representing the external potential, and $f(x, t)$ and $g_{1,\,2}(x, t)$ are the strengths of dispersion and dual-power nonlinearities, respectively. We aim at investigating the integrability and then solve this inhomogeneous NLSE by transforming it into a homogeneous NLSE

$$\begin{aligned} &P(x,t)\Big[\, iU_T(X,T) + A_0\,U_{XX}(X,T) \\ &+ \left(G_{10}\,|U(X,T)|^{2n} + G_{20}\,|U(X,T)|^{4n}\right)U(X,T)\,\Big] = 0, \end{aligned} \tag{2}$$

by applying the following most general form of similarity transformation

$$q(x,t) = U(X,T)\,e^{iA(x,t)+B(x,t)}, \tag{3}$$

where $P(x, t)$, $X(x, t)$, $T(x, t)$, $A(x, t)$, and $B(x, t)$ are real functions and A_0, G_{10}, and G_{20} are real constants. Once these functions are determined, the solutions of Eq. (1) can be obtained from those of Eq. (2) via the transformation (3). Requesting Eq. (1) to transform into Eq. (2), imposes relations among the coefficient functions $f(x, t)$, $g_{1,\,2}(x, t)$, $V(x, t)$, and $W(x, t)$, which will be recognised as *integrability conditions*.

Substituting the transformation (3) in Eq. (1) and requesting the resulting equation to take the form of Eq. (2), we obtain the following conditions:

Coefficient of u_{xx}:

$$-A_0\,P + f\,X_x^2 = 0, \tag{4}$$

Coefficient of g_1:

$$e^{2nB}g_1 - G_{10}\,P = 0, \tag{5}$$