Systems & Control Letters 121 (2018) 1–6

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

An output error bound for time-limited balanced truncation

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ARTICLE INFO

Article history:
Received 20 October 2017
Received in revised form 3 July 2018
Accepted 26 August 2018

Keywords:
Model reduction
Linear systems
Time-limited balanced truncation
Time-limited Gramians
Error bound

ABSTRACT

When solving partial differential equations numerically, usually a high order spatial discretization is needed. Model order reduction (MOR) techniques are often used to reduce the order of spatially-discretized systems and hence reduce computational complexity. A particular MOR technique to obtain a reduced order model (ROM) is balanced truncation (BT). However, if one aims at finding a good ROM on a certain finite time interval only, time-limited BT (TLBT) can be a more accurate alternative. So far, no error bound on TLBT has been proved. In this paper, we close this gap in the theory by providing an output error bound for TLBT with two different representations. The performance of the error bound is then shown in several numerical experiments.

1. Introduction

Let $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times m}$ be a realization of a linear, time-invariant system

$$\boldsymbol{\Sigma}: \quad \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = 0, \quad y(t) = Cx(t) \tag{1}$$

and assume that A is Hurwitz which implies (1) is asymptotically stable. The Hurwitz property is classified by $\Re(\lambda) < 0$ for all $\lambda \in \Lambda(A)$, where $\Lambda(\cdot)$ denotes the spectrum of a matrix.

The infinite reachability and observability Gramians

$$P_\infty = \int_0^\infty e^{As}BB^Te^{A^Ts}ds, \quad Q_\infty = \int_0^\infty e^{A^Ts}C^TCe^{As}ds$$

of (A, B, C) solve the Lyapunov equations

$$AP_\infty + P_\infty A^T + BB^T = 0, \quad A^TQ_\infty + Q_\infty A^T + C^TC = 0. \tag{2}$$

The first ingredient of balanced truncation [1] (BT) is to simultaneously diagonalize both Gramians through congruence transformations $\hat{S}P_\infty\hat{S}^T = \hat{S}^{-T}Q_\infty\hat{S}^{-1} = \Sigma_\infty$ which gives a balanced realization $(\hat{S}A\hat{S}^{-1}, \hat{S}B, C\hat{S}^{-1})$, where Σ_∞ is diagonal and contains the Hankel singular values σ_j (HSVs), i.e., the square roots of the eigenvalues of $P_\infty Q_\infty$. The HSVs σ_j are typically assumed to be ordered in a non-increasing fashion. In the second step the reduced order model $\boldsymbol{\Sigma}_r$ is obtained by keeping only the $r \times r$ upper left block of $\hat{S}A\hat{S}^{-1}$ and the associated parts of $\hat{S}B$, $C\hat{S}^{-1}$, i.e., the smallest $n - r$ HSVs are removed from the system. With Cholesky factorizations $P_\infty = L_PL_P^T$, $Q_\infty = L_QL_Q^T$, and the singular value decomposition (SVD) $X\Sigma_\infty Y^T = L_Q^TL_P$, the balancing transformation is given by $\hat{S} = \Sigma_\infty^{-\frac{1}{2}}X^TL_Q^T$ and $\hat{S}^{-1} = L_PY\Sigma_\infty^{-\frac{1}{2}}$, see, e.g., [2]. Moreover, the resulting reduced system $\boldsymbol{\Sigma}_r$ is asymptotically stable and satisfies the $\mathcal{H}_\infty$ error bound [3]

$$\|\boldsymbol{\Sigma} - \boldsymbol{\Sigma}_r\|_{\mathcal{H}_\infty} \le 2(\sigma_{r+1} + \cdots + \sigma_n). \tag{3}$$

Once the SVD is computed, (3) can be used to adaptively adjust the reduced order r. A generalized $\mathcal{H}_\infty$-error bound for BT has been proved in [4,5], where linear stochastic systems are investigated.

The matrix of truncated HSVs $\Sigma_2 = \operatorname{diag}(\sigma_{r+1}, \ldots, \sigma_n)$ can be used to express the $\mathcal{H}_2$ error bound [2]. It is represented by

$$\|\boldsymbol{\Sigma} - \boldsymbol{\Sigma}_r\|^2_{\mathcal{H}_2} \le \operatorname{tr}(\Sigma_2(B_2B_2^T + 2P_{\infty,M,2}A_{21}^T)), \tag{4}$$

where B_2 is the matrix of the last $n - r$ rows of $\hat{S}B$, A_{21} is the left lower $(n-r)\times r$ block of $\hat{S}A\hat{S}^{-1}$ and $P_{\infty,M,2}$ are the last $n-r$ rows of the mixed Gramian $P_{\infty,M} = \hat{S}\int_0^\infty e^{As}BB_1^Te^{A_{11}^Ts}ds$. The bound in (4) has already been extended to stochastic systems in a more general form [6–8].

In [9] Gawronski and Juang restricted balanced truncation to a finite time interval $[0, \bar{T}]$, $\bar{T} < \infty$, by introducing the time-limited reachability and observability Gramians

$$P_{\bar{T}} := \int_0^{\bar{T}} e^{As}BB^Te^{A^Ts}ds, \quad Q_{\bar{T}} = \int_0^{\bar{T}} e^{A^Ts}C^TCe^{As}ds. \tag{5}$$

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https://doi.org/10.1016/j.sysconle.2018.08.004
0167-6911/

It is easy to show that $P_{\bar{T}},\ Q_{\bar{T}}$ solve the Lyapunov equations

$$AP_{\bar{T}} + P_{\bar{T}}A^T + BB^T - F_{\bar{T}}F_{\bar{T}}^T = 0, \tag{6a}$$

$$A^TQ_{\bar{T}} + Q_{\bar{T}}A^T + C^TC - G_{\bar{T}}^TG_{\bar{T}} = 0, \tag{6b}$$

where $G_t := Ce^{At}$ and $F_t := e^{At}B$, $t \in [0, \bar{T}]$. Time-limited balanced truncation (TLBT) is then carried out by using the Cholesky factors of $P_{\bar{T}}, Q_{\bar{T}}$ instead of P_∞, Q_∞ to construct the balancing transformation which in this case is denoted by S. This transformation simultaneously diagonalizes $P_{\bar{T}}, Q_{\bar{T}}$, i.e., $SP_{\bar{T}}S^T = S^{-T}Q_{\bar{T}}S^{-1} = \Sigma_{\bar{T}}$ and is, thus, referred to as time-limited balancing transformation. The values in $\Sigma_{\bar{T}}$ are referred to as time-limited singular values and are, similar to the HSVs, invariant under state-space transformations. Because of the altered Gramian definitions, TLBT does generally not preserve stability and there is no $\mathcal{H}_\infty$ error bound as in unrestricted BT.

The main contribution of this paper is an output error bound for TLBT. It leads to (4) if $\bar{T} \to \infty$. We provide two representations of this bound. The first one can be used for practical computations and is, hence, an important tool to assess the obtained accuracy. The second representation is not appropriate for computing the bound but it shows that, similar to BT, the time-limited singular values deliver an alternative criterion to find a suitable reduced order dimension r. We conclude this paper by conducting several numerical experiments which indicate that the time-limited error bound is tight.

2. Output error bounds for time-limited balanced truncation

Let S be the time-limited balancing transformation. We partition the balanced realization (SAS^{-1}, SB, CS^{-1}) as follows:

$$SAS^{-1} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad SB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad CS^{-1} = \begin{bmatrix} C_1 & C_2 \end{bmatrix},$$

where $A_{11} \in \mathbb{R}^{r\times r}$, $B_1 \in \mathbb{R}^{r\times m}$, $C_1 \in \mathbb{R}^{p\times r}$ and the other blocks of appropriate dimensions. Furthermore, we introduce

$$SF_{\bar{T}} = \begin{bmatrix} F_{\bar{T},1} \\ F_{\bar{T},2} \end{bmatrix}, \quad G_{\bar{T}}S^{-1} = \begin{bmatrix} G_{\bar{T},1} & G_{\bar{T},2} \end{bmatrix}, \quad \Sigma_{\bar{T}} = \begin{bmatrix} \Sigma_{\bar{T},1} & \\ & \Sigma_{\bar{T},2} \end{bmatrix}.$$

We consider the corresponding Lyapunov equations in partitioned form:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \Sigma_{\bar{T},1} & \\ & \Sigma_{\bar{T},2} \end{bmatrix} + \begin{bmatrix} \Sigma_{\bar{T},1} & \\ & \Sigma_{\bar{T},2} \end{bmatrix} \begin{bmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{bmatrix} \tag{7}$$
$$= -\begin{bmatrix} B_1B_1^T & B_1B_2^T \\ B_2B_1^T & B_2B_2^T \end{bmatrix} + \begin{bmatrix} F_{\bar{T},1}F_{\bar{T},1}^T & F_{\bar{T},1}F_{\bar{T},2}^T \\ F_{\bar{T},2}F_{\bar{T},1}^T & F_{\bar{T},2}F_{\bar{T},2}^T \end{bmatrix},$$

$$\begin{bmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{bmatrix} \begin{bmatrix} \Sigma_{\bar{T},1} & \\ & \Sigma_{\bar{T},2} \end{bmatrix} + \begin{bmatrix} \Sigma_{\bar{T},1} & \\ & \Sigma_{\bar{T},2} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{8}$$
$$= -\begin{bmatrix} C_1^TC_1 & C_1^TC_2 \\ C_2^TC_1 & C_2^TC_2 \end{bmatrix} + \begin{bmatrix} G_{\bar{T},1}^TG_{\bar{T},1} & G_{\bar{T},1}^TG_{\bar{T},2} \\ G_{\bar{T},2}^TG_{\bar{T},1} & G_{\bar{T},2}^TG_{\bar{T},2} \end{bmatrix}.$$

The TLBT reduced system that approximates (1) is given by

$$\dot{x}_r(t) = A_{11}x_r(t) + B_1u(t), \quad x_r(0) = 0, \quad y_r(t) = C_1x_r(t).$$

The goal of this section is to find a bound for the error between y and y_r. Since we have zero initial conditions for both the reduced and the full system, we have the following representations for the outputs

$$y(t) = Cx(t) = C\int_0^t e^{A(t-s)}Bu(s)ds,$$
$$y_r(t) = C_1x_r(t) = C_1\int_0^t e^{A_{11}(t-s)}B_1u(s)ds,$$

where $t \in [0, \bar{T}]$. To find a first representation for the error bound, arguments from [6–8] are used, where a generalized $\mathcal{H}_2$ error bound for stochastic systems has been derived. Some easy rearrangements yield a first error estimate

$$\begin{aligned} &\|y(t) - y_r(t)\|_2 \\ &= \left\| C\int_0^t e^{A(t-s)}Bu(s)ds - C_1\int_0^t e^{A_{11}(t-s)}B_1u(s)ds \right\|_2 \\ &\le \int_0^t \left\| \left(Ce^{A(t-s)}B - C_1e^{A_{11}(t-s)}B_1\right)u(s)\right\|_2 ds \\ &\le \int_0^t \left\| Ce^{A(t-s)}B - C_1e^{A_{11}(t-s)}B_1\right\|_F \|u(s)\|_2\, ds. \end{aligned}$$

By the Cauchy Schwarz inequality it holds that

$$\begin{aligned} &\|y(t) - y_r(t)\|_2 \\ &\le \left(\int_0^t \left\| Ce^{A(t-s)}B - C_1e^{A_{11}(t-s)}B_1\right\|_F^2 ds\right)^{\frac{1}{2}} \left(\int_0^t \|u(s)\|_2^2 ds\right)^{\frac{1}{2}}. \end{aligned}$$

Using substitution, the definition of the Frobenius norm and the linearity of the integral, we obtain

$$\begin{aligned} &\int_0^t \left\| Ce^{A(t-s)}B - C_1e^{A_{11}(t-s)}B_1\right\|_F^2 ds \\ &= \int_0^t \left\| Ce^{As}B - C_1e^{A_{11}s}B_1\right\|_F^2 ds \\ &\le \int_0^{\bar{T}} \left\| Ce^{As}B - C_1e^{A_{11}s}B_1\right\|_F^2 ds \\ &= \int_0^{\bar{T}} \operatorname{tr}\left(Ce^{As}BB^Te^{A^Ts}C^T\right) ds \\ &\quad + \int_0^{\bar{T}} \operatorname{tr}\left(C_1e^{A_{11}s}B_1B_1^Te^{A_{11}^Ts}C_1^T\right) ds \\ &\quad - 2\int_0^{\bar{T}} \operatorname{tr}\left(Ce^{As}BB_1^Te^{A_{11}^Ts}C_1^T\right) ds \\ &= \operatorname{tr}\left(CP_{\bar{T}}C^T\right) + \operatorname{tr}\left(C_1P_{\bar{T},r}C_1^T\right) - 2\ \operatorname{tr}\left(CP_{\bar{T},M}C_1^T\right), \end{aligned}$$

where $P_{\bar{T}} := \int_0^{\bar{T}} e^{As}BB^Te^{A^Ts}ds$, $P_{\bar{T},r} := \int_0^{\bar{T}} e^{A_{11}s}B_1B_1^Te^{A_{11}^Ts}ds$ and $P_{\bar{T},M} := \int_0^{\bar{T}} e^{As}BB_1^Te^{A_{11}^Ts}ds$. Matrix-valued integrals of this form can under some conditions be expressed as unique solutions of matrix equations.

Lemma 2.1. *Let $A_1 \in \mathbb{R}^{n\times n}$, $A_2 \in \mathbb{R}^{r\times r}$ with $\Lambda(A_1) \cap -\Lambda(A_2) = \emptyset$ and $B_1 \in \mathbb{R}^{n\times m}$, $B_2 \in \mathbb{R}^{r\times m}$. Then,*

$$X = \int_0^{\bar{T}} e^{A_1s}B_1B_2^Te^{A_2^Ts}ds$$

solves the Sylvester equation

$$A_1X + XA_2^T = -B_1B_2^T + e^{A_1\bar{T}}B_1B_2^Te^{A_2^T\bar{T}}.$$

Proof. The integral is equivalent to

$$\begin{aligned} \operatorname{vec}(X) &= \int_0^{\bar{T}} \operatorname{vec}\left(e^{A_1s}B_1B_2^Te^{A_2^Ts}\right) ds \\ &= \int_0^{\bar{T}} e^{A_2s} \otimes e^{A_1s} ds \operatorname{vec}\left(B_1B_2^T\right) \\ &= \int_0^{\bar{T}} e^{(I_r\otimes A_1 + A_2\otimes I_n)s} ds \operatorname{vec}\left(B_1B_2^T\right), \end{aligned}$$