Journal of Computational Physics 376 (2019) 98–128

Contents lists available at ScienceDirect

Journal of Computational Physics

www.elsevier.com/locate/jcp

Direct implementation of high order BGT artificial boundary conditions ☆

M. Medvinsky [a], S. Tsynkov [a,*], E. Turkel [b]

[a] Department of Mathematics, North Carolina State University, Box 8205, Raleigh, NC 27695, USA
[b] School of Mathematical Sciences, Tel Aviv University, Ramat Aviv, Tel Aviv 69978, Israel

ARTICLE INFO

Article history:
Received 3 May 2018
Received in revised form 18 September 2018
Accepted 18 September 2018
Available online xxxx

Keywords:
Propagation of waves over unbounded regions
Helmholtz equation
Spherical artificial boundary
Expansion with respect to continuous basis
Method of difference potentials (MDP)
Beltrami operator

ABSTRACT

Local artificial boundary conditions (ABCs) for the numerical simulation of waves have been successfully used for decades (most notably, the boundary conditions due to Engquist & Majda, Bayliss, Gunzburger & Turkel, and Higdon). The basic idea behind these boundary conditions is that they cancel several leading terms in an expansion of the solution. The larger the number of terms canceled, the higher the order of the boundary condition and, in turn, the smaller the reflection error due to truncation of the original unbounded domain by an artificial outer boundary. In practice, however, the use of local ABCs has been limited to low orders (first and second), because higher order boundary conditions involve higher order derivatives of the solution, which may harm well-posedness and cause numerical instabilities. They are also difficult to implement especially in finite elements. A prominent exception is the development of local high order ABCs based on auxiliary variables.

In the current paper, we implement high order Bayliss–Turkel ABCs directly — with no auxiliary variables yet no discrete approximation of the constituent high order derivatives either. Instead, we represent the solution at the boundary as an expansion with respect to a continuous basis. For the spherical artificial boundary, the basis consists of eigenfunctions of the Beltrami operator (spherical harmonics), which enable replacing the high order derivatives in the ABCs with powers of the corresponding eigenvalues. The continuous representation at the boundary is coupled to higher order compact finite differences inside the domain by the method of difference potentials (MDP). It maintains high order accuracy even when the boundary is not aligned with the discretization grid.

1. Introduction

Describing the behavior of scattered waves about a body is essential in many fields, whether it be the reflection of sonar waves from a submarine, the reflection of radar waves from an airplane or the reflection of microwaves from a

☆ Work supported by the US Army Research Office (ARO) under grant # W911NF-16-1-0115, and by the US-Israel Binational Science Foundation (BSF) under grant # 2014048.
* Corresponding author.
E-mail addresses: mmedvin@ncsu.edu (M. Medvinsky), tsynkov@math.ncsu.edu (S. Tsynkov), turkel@post.tau.ac.il (E. Turkel).
URLs: https://mmedvin.math.ncsu.edu (M. Medvinsky), https://stsynkov.math.ncsu.edu (S. Tsynkov), http://www.math.tau.ac.il/~turkel/ (E. Turkel).

https://doi.org/10.1016/j.jcp.2018.09.040
0021-9991/

cellular phone. In computing such solutions, one, in principle, needs to compute on an unbounded domain. Analytically, the Sommerfeld condition allows only outgoing waves and eliminates those coming in from infinity.

Unfortunately, due the finiteness of the computer it is not feasible to compute on an unbounded domain. There are several approaches to overcoming this difficulty. The approach we use develops a finite difference (or finite element) approximation, which is grid based. One then adds an artificial outer surface. On this surface, an artificial boundary condition (ABC) is imposed that reduces the reflection of waves back into the domain of interest. Another approach would be to reformulate the partial differential equation as a boundary integral equation. This integral equation contains a Green's function which automatically allows only the correct behavior at infinity. Yet another approach would be to use finite elements that are unbounded (the so-called infinite elements [1]).

There are three main methods for the treatment of artificial outer boundaries for wave-like equations. Early techniques used local approximations. In particular, the approach by Engquist and Majda [2] relies on splitting the time-dependent equation into forward and backward parts and using pseudo-differential analysis to gain approximate ABCs. The approach by Higdon [3,4] makes the artificial boundary transparent for plane waves traveling at predetermined incidence angles. The approach by Bayliss and Turkel [5,6] is based on expansions in an inverse radius for both the time-dependent and time-harmonic cases. To improve the accuracy of local techniques, one can use non-local operators that may employ an integral or pseudo-differential formulation along the boundary. This is frequently expressed as a Dirichlet to Neumann map (DtN) [7–9]. An alternative to both local and non-local ABCs is provided by the approach initiated by Bérenger [10]. It constructs an exterior layer to the domain that matches the interior in a way to greatly reduce the reflections. This is usually refereed to as a perfectly matched layer (PML).

In this work, we concentrate on the Bayliss–Gunzburger–Turkel (BGT) approach [6] for the 3D Helmholtz equation. The BGT algorithm constructs a sequence of increasingly more accurate ABCs; the accuracy increases as the spherical radius r increases, the wavenumber k increases, and the order of the ABC increases. In practice, however, only the first two BGT operators are used since higher order operators require high order tangential derivatives. This is especially difficult for finite element methods but is also a difficulty for finite difference methods. A similar difficulty affects the Engquist–Majda and Higdon sequences of operators. One way to overcome this difficulty is to add auxiliary variables instead of employing a larger stencil. This approach has been applied to the Engquist–Majda, Higdon, and BGT sequences of ABCs [11–17]. The Engquist–Majda and Higdon ABCs are most straightforward to use in Cartesian coordinates, while the BGT ABCs are easier to implement in polar or spherical coordinates. Zarmi and Turkel [18] consider a more general way of extending the BGT approach to other coordinate systems.

For wave-like equations it is well known that both finite differences and finite elements suffer from a pollution effect. This implies that the numerical error grows faster than linear with the frequency of the wave. This growth decreases as the rate of accuracy of the scheme increases. Hence, it is important to solve wave propagation problems with high order schemes. In practice, a fourth order or possibly sixth order accurate scheme is sufficient. In this paper, we solve the frequency domain wave equation, i.e., the Helmholtz equation, in three space dimensions using a sixth order accurate compact finite difference scheme [19].

The total error is therefore composed of two separate entities: the discretization error of the scheme in the interior of the domain and the error caused by reflections from the artificial outer boundary. In general, the global error will be dominated by the larger of these errors. Hence, it is important to reduce both sets of errors. As stated above, we will use a sixth order accurate compact finite difference scheme for the Helmholtz equation. In addition, we need to reduce the error from the BGT boundary condition, i.e., increase its order. In doing so, we would like to avoid using large stencils for the approximation of high order derivatives, as this is not practical.

Therefore, in this paper we introduce a new method of imposing high order BGT boundary conditions. It consists of first developing a sequence of explicit formulae for the ABC operators that involve powers of the Beltrami operator on the spherical artificial boundary. Then, the solution at the boundary is expanded in the basis of spherical harmonics that are eigenfunctions of the Beltrami operator. This reduces powers of the Beltrami operator in the ABCs to powers of the corresponding eigenvalues, and hence requires neither the approximation of high order tangential derivatives on large stencils nor the introduction of auxiliary variables. Finally, the BGT condition represented in the basis of spherical harmonics at the boundary is coupled with a sixth order finite difference scheme that approximates the Helmholtz equation on the domain via the method of difference potentials (MDP) [20]. The MDP enables high order accuracy even for non-conforming boundaries. In our case, the artificial boundary is spherical and the discretization grid is Cartesian.

Computational experiments are then performed to show the power of the new approach. As expected, both the wavenumber and the position of the artificial outer surface (i.e., the radius of the spherical domain) affect the accuracy of the ABC and thus the balance between the discretization and reflection error. As the outer surface is brought closer in (reducing the computational effort) or the wavenumber is reduced, there is a greater need for a high order ABC.

2. BGT boundary conditions

We derive convenient explicit formulae for the three dimensional BGT artificial boundary conditions. The high order tangential derivatives in the ABCs appear as powers of the spherical Beltrami operator. These formulae are of use by them-