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Numerical solution of the electron transport equation in the upper atmosphere ☆

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A B S T R A C T

A new approach for solving the electron transport equation in the upper atmosphere is derived. The problem is a very stiff boundary value problem, and to obtain an accurate numerical solution, matrix factorizations are used to decouple the fast and slow modes. A stable finite difference method is applied to each mode. This solver is applied to a simplified problem for which an exact solution exists using various versions of the boundary conditions that might arise in a natural auroral display. The numerical and exact solutions are found to agree with each other, verifying the method.

1. Introduction

The principle cause of the aurora is charged particles from the sun interacting with the earth's upper atmosphere. During a solar storm, the sun releases charged particles called the solar wind, which consists mostly of electrons and protons. Some of these particles reach the earth where they accelerate along the magnetic field lines which converge at the poles. Once the particles reach sufficiently low altitudes they scatter off atmospheric atoms and molecules. This scattering causes the atoms and molecules to enter excited states, and they return to their ground states via collisional quenching and fluorescence. Most of the auroral light is due to electron impact excitations, which is why we focus on electron transport. A more comprehensive explanation of how this occurs can be found in Rees [1].

The electron transport problem can be modeled as an integro-differential equation. What is challenging about the problem is the unusual way the boundary conditions are split between the upper and lower limits of the atmosphere, and the pronounced boundary-layer structure of the solution due to the stiffness of the equation. Both Monte Carlo and deterministic methods have been used, but only the latter are more relevant to our study. The most well-known approach is due to Stamnes et al. [2]. This method subdivides the atmosphere into layers, and in each layer the problem is replaced by one with constant coefficients. The source term is approximated by an exponential, and the system of equations is solved exactly within each layer. These layers are patched together by requiring the solution to be continuous. This method has been used

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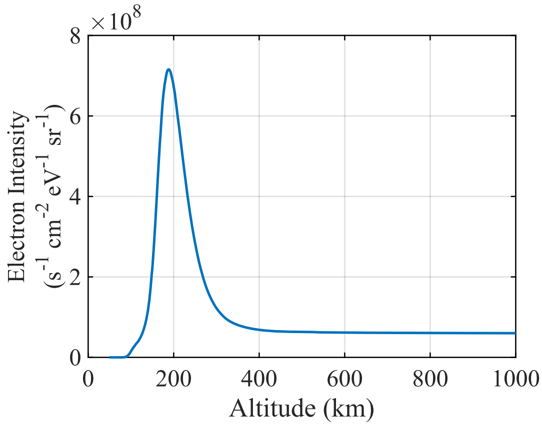

Fig. 1. An example electron intensity solution at 1.2 eV.

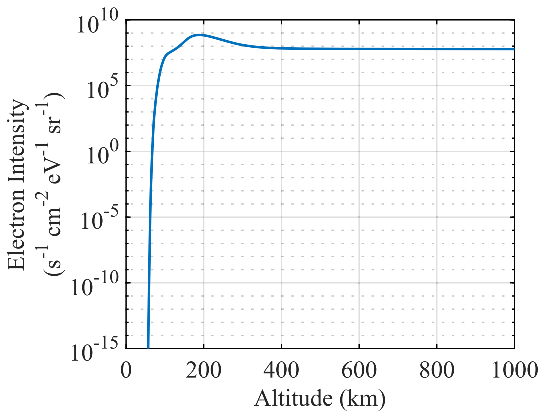

Fig. 2. The same intensity on a logarithmic scale.

in Stamnes [3], Lummerzheim et al. [4], Min et al. [5], Lummerzheim and Lilensten [6], and most recently in Lanchester and Gustavsson [7].

What has been missed in the above and other numerical methods is an accurate accounting of the lower atmosphere. To explain, at the upper boundary a downward electron distribution is specified. Similarly, at some low altitude the upward electron distribution is set to zero. The top of the upper atmosphere is simply chosen to be an altitude where the density is relatively small and scattering effects are negligible. The bottom of the atmosphere is more troublesome. Theoretically, the ground (an altitude of zero) could be chosen because there are no free electrons at ground level. However, the electron transport equation has the property of becoming exponentially stiff at lower altitudes. This region can be avoided by prescribing a lower boundary that is far from ground level, but this yields a model which is not physically meaningful. On the other hand, if a more realistic lower boundary is used then standard numerical methods, such as collocation at Gaussian or Lobatto points, will produce negative and oscillating solutions due to the exponential stiffness.

Regardless of what low altitude is chosen, the solution must smoothly approach zero as the altitude decreases. Typical auroral electron intensities approach zero somewhere below 100 km where the exact altitude is dependent on the strength of the solar wind. This is exactly the region where the problem is exponentially stiff. A strong solar wind can give electron intensities on the order of $10^8\, e^-\ \mathrm{s}^{-1}\,\mathrm{cm}^{-2}\,\mathrm{eV}^{-1}\,\mathrm{sr}^{-1}$ or more (see Arnoldy and Lewis [8] and Solomon et al. [9]). This means that the numerical solution of the electron transport equation must drop many orders of magnitude over a few dozen km. The very rapid drop in intensity as the electrons approach the lower altitudes causes most solvers, even those designed for stiff problems, to overshoot or oscillate. The resulting negative intensities are physically meaningless, and the problem is unstable in such cases. In Figs. 1 and 2, we show an example of downward streaming electron intensities for 1.2 eV electrons (the curves come from solutions to a numerical example given in Section 5). This energy was chosen because there is a greater abundance of low energy electrons, so the change in scale is readily apparent. The point, however, is that the numerical solution should approach zero as altitude decreases without overshoot or oscillation.