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Three recipes for quasi-interpolation with cubic Powell–Sabin splines ☆

Jan Grošelj [a], Hendrik Speleers [b,*]

[a] Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia
[b] Department of Mathematics, University of Rome Tor Vergata, Via della Ricerca Scientifica, 00133 Rome, Italy

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ABSTRACT

We investigate the construction of bivariate quasi-interpolation methods based on C^1 cubic Powell–Sabin B-spline representations. Rather than using a large set of functional data to specify all the parameters in such representations, we study how to reduce them by imposing different super-smoothness properties while retaining cubic precision. This results in three recipes, which are completely general in the sense that they can be implemented with any local cubic polynomial approximation scheme (or a mixture of them). More precisely, they embed C^2 super-smoothness at the vertices and across the edges, C^2 super-smoothness inside the macro-triangles, and smoothness of Clough–Tocher type, respectively. To demonstrate their usefulness, we derive four specific methods based on local Hermite and Lagrange interpolation. We conclude with a selection of numerical experiments.

1. Introduction

Quasi-interpolation is a term describing approximation methods based on local approximations of functional data. Such methods often make use of standard approximation techniques (e.g., polynomial interpolation and approximation in the least square sense) on data corresponding to small parts of the domain and then blend the results into a global approximation scheme. This approach ensures locality of the scheme and avoids costly computations on a global level.

A common way to define a quasi-interpolant is by representing it in terms of a polynomial spline expressed in a certain basis and then utilizing local approximations to specify the coefficients of the spline. In this setting, the approximation properties of the quasi-interpolant depend on the quality of both the basis functions and the local approximation operators that determine the coefficients. Typically, basis functions of choice are locally supported (this limits the influence of local operators to designated regions) and form a convex partition of unity (this implies computational stability). The local approximation operators, on the other hand, should be defined in such a way that the quasi-interpolant reproduces polynomials to the highest possible degree (this usually ensures optimal approximation order). There is a vast literature explaining these concepts in more detail; see, e.g., Chui (1988), de Boor (2001), Sablonnière (2005) and references therein.

In this paper we focus on bivariate quasi-interpolation methods based on polynomial splines on triangulations. Setting up a suitable basis on general triangulations has shown to be an extremely difficult problem, emerging from the fact that

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* Corresponding author.
E-mail addresses: jan.groselj@fmf.uni-lj.si (J. Grošelj), speleers@mat.uniroma2.it (H. Speleers).

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the dimension of a spline space depends on the geometry of the underlying triangulation. Hence, a great deal of methods is associated with spline bases on triangulations of very special types, the most prominent example of which are the box-splines defined on three-directional meshes (see de Boor et al., 1993). An alternative approach enabling the construction of quasi-interpolants on general triangulations is to use macro-element spaces, i.e., spline spaces that are defined on special refinements of the original triangulations (see Lai and Schumaker, 2007).

Several splitting techniques can be used to obtain a spline space of particular degree and smoothness. However, the most versatile to construct suitable basis functions may be the Powell–Sabin technique that splits every triangle of the original triangulation into six smaller parts (Powell and Sabin, 1977). This is the merit of an intuitive geometric construction of B-spline like functions developed by Dierckx (1997) that was first applied to the standard Powell–Sabin C^1 spline space of degree two and later generalized to many different super-spline spaces covering arbitrary degree and smoothness (see, e.g., Grošelj, 2016; Grošelj and Krajnc, 2016; Grošelj and Speleers, 2017; Lamnii et al., 2014b; Speleers, 2013, 2015b). Such B-spline representations on Powell–Sabin triangulations have been employed in a variety of quasi-interpolation methods, ranging from those defined in the context of standard C^1 quadratic splines (Manni and Sablonnière, 2007; Sbibih et al., 2009) to more complex ones possessing a higher order of smoothness (Speleers, 2012, 2015a). A considerable attention has also been dedicated to C^1 cubic quasi-interpolants with special super-smoothness properties (Lamnii et al., 2014a; Sbibih et al., 2015).

The aim of this paper is to provide a general framework of quasi-interpolation methods based on the cubic B-splines recently proposed by Grošelj and Speleers (2017). These functions form a basis of the full C^1 cubic spline space on a Powell–Sabin triangulation. This means that the discussed quasi-interpolants have plenty of parameters and also cover previously considered quadratic and cubic quasi-interpolants. Rather than using a great extent of functional data to specify these parameters, we study how to reduce them by imposing different super-smoothness properties while retaining cubic precision. Throughout the paper we establish three general recipes that can be used to define:

- quasi-interpolants with additional C^2 smoothness at the vertices and across the edges of the triangulation,
- quasi-interpolants with additional C^2 smoothness inside the macro-triangles of the triangulation, and
- quasi-interpolants of Clough–Tocher type.

The recipes are completely general in the sense that they can be implemented with any local cubic polynomial approximation scheme, or even a mixture of such local approximation schemes. To demonstrate the usefulness of these recipes, we derive four specific methods based on local Hermite and Lagrange interpolation. Three of these four methods have not been considered before in the literature, and our numerical experiments show that they have some advantages over the existing one.

The remainder of the paper is organized as follows. In Section 2 we provide some preliminaries on Bernstein–Bézier methods for splines on triangulations and review the B-spline representation of C^1 cubic Powell–Sabin splines. In Section 3 the main results of the paper are stated. We introduce a quasi-interpolant in its most general form and then investigate how to reduce its degrees of freedom. After providing three basic recipes we present the conditions to ensure cubic precision. In Section 4 we derive four practical quasi-interpolants and compare them numerically in terms of approximation quality and functional evaluations. We conclude with some remarks in Section 5.

2. A C^1 cubic B-spline representation

In this section we first provide some preliminaries on polynomials on triangles and splines on triangulations. Then, we review the properties of the B-spline representation of C^1 cubic Powell–Sabin splines studied by Grošelj and Speleers (2017). Every such spline s is defined on a Powell–Sabin refinement $\triangle_{\mathrm{PS}}$ of a triangulation $\triangle$ (which splits every triangle of $\triangle$ into six smaller triangles) and can be represented uniquely in the form

$$s = \sum_{i=1}^{|\mathcal{V}|} \sum_{r=1}^{3} c_{i,r}^{v} B_{i,r}^{v} + \sum_{m=1}^{|\triangle_{\mathrm{PS}}|} c_m^t B_m^t + \sum_{\ell=1}^{|\mathcal{E}_{\mathrm{PS}}^b|} c_\ell^e B_\ell^e \tag{1}$$

for some real values $c_{i,r}^v$, c_m^t, c_ℓ^e. The basis functions $B_{i,r}^v$, B_m^v, B_ℓ^e are called B-splines since they form a convex partition of unity. There are three B-splines $B_{i,r}^v$, $r = 1, 2, 3$, associated with every vertex $\boldsymbol{V}_i$ of $\triangle$, where the index i takes the values from 1 to the number $|\mathcal{V}|$ of vertices $\mathcal{V}$ of $\triangle$. Furthermore, there is one B-spline B_m^t associated with every triangle $\mathcal{T}_m$ of $\triangle_{\mathrm{PS}}$, where the index m takes the values from 1 to the number $|\triangle_{\mathrm{PS}}|$ of triangles in $\triangle_{\mathrm{PS}}$. Finally, there is one additional B-spline B_ℓ^e associated with every boundary edge ε_ℓ of $\triangle_{\mathrm{PS}}$, where the index ℓ takes the values from 1 to the number $|\mathcal{E}_{\mathrm{PS}}^b|$ of boundary edges $\mathcal{E}_{\mathrm{PS}}^b$ of $\triangle_{\mathrm{PS}}$.

2.1. Splines on triangulations

Suppose that $\triangle$ is a triangulation of a polygonal domain $\Omega \subset \mathbb{R}^2$. The space $\mathbb{S}_d^r(\triangle)$ of C^r polynomial splines of degree d on $\triangle$ is defined by