# On the weak tightness and power homogeneous compacta

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A R T I C L E   I N F O

A B S T R A C T

Motivated by results of Juhász and van Mill in [13], we define the cardinal invariant $wt(X)$, the weak tightness of a topological space $X$, and show that $|X| \leq 2^{L(X)wt(X)\psi(X)}$ for any Hausdorff space $X$ (Theorem 2.8). As $wt(X) \leq t(X)$ for any space $X$, this generalizes the well-known cardinal inequality $|X| \leq 2^{L(X)t(X)\psi(X)}$ for Hausdorff spaces (Arhangel'skiǐ [1], Šapirovskiǐ [17]) in a new direction. Theorem 2.8 is generalized further using covers by $G_\kappa$-sets, where $\kappa$ is a cardinal, to show that if $X$ is a power homogeneous compactum with a countable cover of dense, countably tight subspaces then $|X| \leq \mathfrak{c}$, the cardinality of the continuum. This extends a result in [13] to the power homogeneous setting.

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## 1. Introduction

In 2006 R. de la Vega [10] answered a long-standing question of A.V. Arhangel'skiǐ by showing that the cardinality of any homogeneous compactum is at most $2^{t(X)}$, where the cardinal invariant $t(X)$ is the *tightness* of $X$ (Definition 2.1). This was a landmark result in the theory of homogeneous topological spaces and has motivated subsequent work and generalizations. Recall that a space $X$ is *homogeneous* if for all $x, y \in X$ there exists a homeomorphism $h : x \to y$ such that $h(x) = y$. Roughly, a space is homogeneous if all points in the space share identical topological properties. A space is *power homogeneous* if there exists a cardinal $\kappa$ such that $X^\kappa$ is homogeneous.

Recently, Juhász and van Mill [13] introduced fundamentally new techniques and gave the following intricate improvement of De la Vega's theorem. Their result generalizes that theorem in the countably tight case.

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**Theorem 1.1** *(Juhász, van Mill [13]). If a compactum $X$ is the union of countably many dense countably tight subspaces and $X^\omega$ is homogeneous, then $|X| \leq \mathfrak{c}$.*

It was observed by Arhangel′skiǐ in [3] that De la Vega's theorem in fact follows from the following theorem of Pykeev involving the $G_\kappa$-*modification* $X_\kappa$ of a space $X$, the space formed on the same underlying set as $X$ with topology generated by the $G_\kappa$-sets. (See Definition 3.1.)

**Theorem 1.2** *(Pytkeev, [14]). If $X$ is compact and $\kappa$ a cardinal, then $L(X_\kappa) \leq 2^{t(X) \cdot \kappa}$.*

A short, direct proof of De la Vega's theorem using Pytkeev's theorem was given in [8]. In addition, a generalization of Pytkeev's theorem proved in that paper has as a short corollary the well-known cardinality bound $2^{L(X)t(X)\psi(X)}$ for the cardinality of any Hausdorff space $X$ (Arhangel′skiǐ [1], Šapirovskiǐ [17]). This established that the $G_\kappa$-modification of a space can be used to derive related cardinality bounds for both general spaces and homogeneous spaces simultaneously. These related bounds may be thought of as "companion" bounds. In the compact setting, De la Vega's theorem may be thought of as the homogeneous companion bound to the cardinality bound $2^{\chi(X)}$ for general compacta. Both follow from Pytkeev's theorem. Results in a similar vein occur in [5], [6], [7], [9], [18], and elsewhere.

In this study we develop the companion bound for general spaces to Theorem 1.1 above. To this end we introduce the *weak tightness* $wt(X)$ of a space (Definition 2.3), which satisfies $wt(X) \leq t(X)$, and prove the following theorem and central result in this paper. For a space $X$, the $G_\kappa^c$-modification $X_\kappa^c$ of $X$ is a variation of $X_\kappa$ (see Definition 3.1). Note that if $X$ is regular then $X_\kappa^c \approx X_\kappa$.

**Main Theorem.** *For any space $X$ and infinite cardinal $\kappa$, $L(X_\kappa^c) \leq 2^{L(X)wt(X) \cdot \kappa}$.*

From the above theorem several corollaries follow. First, one corollary is that $|X| \leq 2^{L(X)wt(X)\psi(X)}$ for any Hausdorff space $X$ (Theorem 2.8). This improves the Arhangel′skiǐ–Šapirovskiǐ result. A second corollary is an extension of the Juhász-van Mill result 1.1 into the power homogeneous setting. We show in Theorem 4.6 that if $X$ is power homogeneous compactum that is the countable union of countably tight and dense in $X$ then $|X| \leq \mathfrak{c}$. This also generalizes the following theorem in [4]: if $X$ is a power homogeneous, countably tight compactum then $|X| \leq \mathfrak{c}$.

Third, a result of Bella and Spadaro from [7] follows as a corollary to the Main Theorem and can be strengthened to the power homogeneous setting. In that paper Theorem 1.1 was extended using the $G_\delta$-modification $X_\delta$ by showing that if $X$ is a homogeneous compactum which is the union of countably my dense countably tight subspaces, then $L(X_\delta) \leq \mathfrak{c}$. Using the Main Theorem, we show in Theorem 4.7 that this result holds even if the homogeneity property is replaced with the weaker power homogeneity property.

Finally, observe that it follows from the Main Theorem that if $X$ is compact and $\kappa$ a cardinal, then $L(X_\kappa^c) \leq 2^{wt(X) \cdot \kappa}$. This is a strengthening of Pytkeev's theorem.

This paper is organized as follows. In §2 we define the cardinal invariant $wt(X)$ and give a direct proof that if $X$ is Hausdorff then $|X| \leq 2^{L(X)wt(X)\psi(X)}$ in Theorem 2.8. This uses the fact that if $X$ is Hausdorff and $D \subseteq X$, then $|\overline{D}| \leq |D|^{wt(X)\psi_c(X)}$ (Theorem 2.5). In §3 we prove the full generalization of Theorem 2.8, the Main Theorem mentioned above. Corollaries involving compact power homogeneous spaces follow in §4.

For all undefined notions we refer the reader to [11] and [12]. We make no implicit assumptions concerning separation axioms, although by *compactum* we will mean a compact, Hausdorff space.

## 2. An improved cardinality bound for Hausdorff spaces

In this section we give a direct proof of an improved cardinality bound for Hausdorff spaces. We begin with three definitions. First, we recall the definition of the tightness $t(X)$ of a space $X$. Observe any first countable space is countably tight.