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Nonblockers in homogeneous continua

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ABSTRACT

Given a continuum X, an element $A \in 2^X \setminus \{X\}$ is said to be a set that does not block singletons of X provided that for every point $x \in X \setminus A$, there exists an order arc $\alpha : [0,1] \to C(X)$ such that $\alpha(0) = \{x\}$, $\alpha(1) = X$ and $\alpha(t) \cap A = \emptyset$ for all $t \in [0,1)$. If A does not block singletons of X, we say that A is a *nonblocker* of X. In this paper, we present some properties about the sets that do not block singletons and, for a homogeneous curve X, we use the set $\mathcal{NB}(\mathcal{F}_1(X))$ to construct a continuous decomposition of X that is a homogeneous curve and a Whitney level of $C(X)$, where its elements are homogeneous, acyclic, hereditarily unicoherent and indecomposable continuum. Also, we prove that if X is a continuum such that $\mathcal{NB}(\mathcal{F}_1(X)) = \mathcal{F}_1(X)$, then the next three properties are equivalent: 1) X is a homogeneous continuum, 2) X has the property that for all $x \in X$, there exists a proper subcontinuum A of X such that $x \in Int(A)$, and 3) $X \approx \mathcal{S}^1$.

1. Introduction

In this paper we study the concept of nonblocker of a continuum X and the hyperspace that consists of all the nonblockers of X (defined in [3] and denoted by $\mathcal{NB}(\mathcal{F}_1(X))$). In the Section 3, we develop some general results concerning to nonblockers of X. In Section 4, we present some results concerning to the study of $\mathcal{NB}(\mathcal{F}_1(X))$ of homogeneous continua, particularly we show results about continuous decomposition of homogeneous curves X generated by the set $\mathcal{NB}(\mathcal{F}_1(X))$ (Corollary 4.5).

Finally, in Theorem 4.4 of [4] the authors prove that if X is locally connected, then $\mathcal{NB}(\mathcal{F}_1(X)) = \mathcal{F}_1(X)$ if and only if $X \approx S^1$. In Section 5, we reduce the hypothesis used in Theorem 4.4 of [4] and obtain other equivalences presented in Theorem 5.4.

2. Definitions and notation

Given a subset A of a metric space Z with metric d, the closure, the boundary and the interior of A are denoted by $Cl_Z(A)$, $Bd_Z(A)$ and $Int_Z(A)$, respectively. Also, $\mathcal{V}_r(A)$ denotes the open ball of radius r

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about A. A *map* is a continuous function. A *continuum* is a nonempty compact connected metric space. A continuum X is *aposyndetic at* p *with respect to* q, where $p, q \in X$, provided that there is a subcontinuum A of $X \setminus \{q\}$ such that $p \in Int_X(A)$. A continuum X is *aposyndetic at* p provided that X is aposyndetic at p with respect to each point $q \in X \setminus \{p\}$. A continuum is *homogeneous* if for any two points p and q of X, there exists a homeomorphism h from X to itself such that $h(p) = q$. A continuum X is *unicoherent* provided that whenever A and B are closed connected subsets of X such that $X = A \cup B$, then $A \cap B$ is connected. A continuum X is *indecomposable* provided that X cannot be written as the union of two of its proper subcontinua. A set Y is *continuumwise connected* if any pair of points in Y belong to a subcontinuum of Y. A *curve* is a one dimensional continuum. $\mathcal{S}^1 = \{x \in \mathbb{R}^2 : ||x|| = 1\}$. For any nondegenerate continuum X and any point $p \in X$, let

$$k(p) = \{x \in X : \text{ there is a proper subcontinuum } A \text{ of } X \text{ such that } p, x \in A\}.$$

The sets $k(p)$ are called *composants* of X.

Given a continuum X, we consider the following *hyperspaces* of X:

$$2^X = \{A \subset X : A \text{ is nonempty and closed}\},$$
$$\mathcal{C}_n(X) = \{A \in 2^X : A \text{ has at most } n \text{ components}\}$$

and

$$\mathcal{F}_n(X) = \{A \in \mathcal{C}(X) : A \text{ has at most } n \text{ elements}\},$$

where n is a positive integer. $\mathcal{C}_n(X)$ and $\mathcal{F}_n(X)$ are called the *n-fold hyperspace* of X and the *n-fold symmetric product* of X respectively. These spaces are topologized with the Hausdorff metric defined as follows:

$$\mathcal{H}(A, B) = \inf \{\epsilon > 0 : A \subset \mathcal{V}_\epsilon(B) \text{ and } B \subset \mathcal{V}_\epsilon(A)\},$$

$\mathcal{H}$ always denotes the Hausdorff metric on 2^X. When $n = 1$, we write $\mathcal{C}(X)$ instead of $\mathcal{C}_1(X)$.

Clearly $\mathcal{F}_n(X) \subset \mathcal{C}_n(X) \subset 2^X$ and $\mathcal{F}_1(X) \approx X$. It is known that if X is a continuum then 2^X, $\mathcal{C}(X)$ (Theorem 1.13 of [10]) and $\mathcal{C}_n(X)$ (Corollary 1.8.12 of [7]) are arcwise connected continua. Also, $\mathcal{F}_n(X)$ is a continuum for all positives integers n (p. 887 of [2]).

Let $X_1, X_2, \ldots, X_m$ be a finite collection of subsets of X. We define the subset $\langle X_1, X_2, \ldots, X_m\rangle_n$ of $\mathcal{C}_n(X)$ by $\langle X_1, X_2, \ldots, X_m\rangle_n = \{A \in \mathcal{C}_n(X) : A \subset X_1 \cup \ldots \cup X_m \text{ and } A \cap X_k \neq \emptyset \text{ for } k = 1, \ldots, m\}$.

It is known that if $X_1, X_2, \ldots, X_m$ are closed subsets of X, then $\langle X_1, X_2, \ldots, X_m\rangle_n$ is closed in $\mathcal{C}_n(X)$ and that the collection of all subsets of $\mathcal{C}_n(X)$ of the form $\langle U_1, U_2, \ldots, U_m\rangle_n$, where $U_1, U_2, \ldots, U_m$ are open subsets of X, is a base for the topology of $\mathcal{C}_n(X)$ (see [6]).

A *path* in 2^X from A to B is a map $\gamma : [0, 1] \to 2^X$ such that $\gamma(0) = A$ and $\gamma(1) = B$. An *order arc* in 2^X is an arc $\alpha : [0, 1] \to 2^X$ such that if $0 \leq s < t \leq 1$, then $\alpha(s) \subset \alpha(t)$ and $\alpha(s) \neq \alpha(t)$. By Lemma 1.11 of [10], if $\alpha(0) \in \mathcal{F}_1(X)$, then $\alpha([0, 1]) \subset \mathcal{C}(X)$.

A map $\mu : \mathcal{C}(X) \to [0, \infty)$ is said to be a *Whitney map* provided that:

1. $\mu(A) = 0$ for every $A \in \mathcal{F}_1(X)$;
2. if $A \subset B$ and $A \neq B$, then $\mu(A) < \mu(B)$.

Since X is nondegenerate, we may assume that $\mu(X) = 1$. By (0.50.3) of [10], Whitney maps exist for each continuum X.