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Whitehead torsion of inertial h-cobordisms ☆

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ABSTRACT

We study the Whitehead torsion of inertial h-cobordisms, continuing an investigation started in [12]. Of particular interest is a nested sequence of subsets of the Whitehead group, and a number of examples are given to show that these subsets are all different in general. The main new results are Theorems 2.5, 2.6 and 2.7. Proposition 5.2 is a partial correction to [12, Lemma 8.1].

1. Introduction

The h-cobordism theorem plays a crucial role in modern geometric topology, providing the essential link between homotopy and geometry. Indeed, comparing manifolds of the same homotopy type, one can often use surgery methods to produce h-cobordisms between them, and then hope to be able to show that the Whitehead torsion $\tau(W^{n+1}; M^n)$ in $\mathrm{Wh}(\pi_1(M^n))$ is trivial. By the s-cobordism theorem, the two manifolds will then be isomorphic (homeomorphic or diffeomorphic, according to which category we work in).

The last step, however, is in general very difficult, and what makes the problem even more complicated, but at the same time more interesting, is that there exist h-cobordisms with non-zero torsion, but where the ends still are isomorphic (cf. [8], [9], [18], [12]). Such h-cobordisms are called *inertial*. The central problem is then to determine the subset of elements of the Whitehead group $\mathrm{Wh}(\pi_1(M^n))$ which can be realized as

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Whitehead torsion of inertial h-cobordisms of the manifold M. This is in general very difficult, and only partial results in this direction are known ([8], [9], [18]).

The purpose of this note is to shed some light on this important problem.

2. Inertial h-cobordisms

In this section we recall basic notions and constructions concerning various types of h-cobordisms. We will follow the notation and terminology of [12]. For convenience we choose to formulate everything in the category of topological manifolds, but for most of what we are going to say, this does not make much difference. See Section 5 for more on the relations between the different categories.

An h-cobordism $(W; M, M')$ is a compact manifold W with two boundary components M and M', each of which is a deformation retract of W.

We will think of this as an h-cobordism from M to M', thus distinguishing it from the dual h-cobordism $(W; M', M)$. Since the pair $(W; M)$ determines M', we will often use the notation $(W; M)$ for $(W; M, M')$. We denote by $\mathcal{H}(M)$ the set of homeomorphism classes relative M of h-cobordisms from M.

If X is a path connected space, we denote by $\mathrm{Wh}(X)$ the Whitehead group $\mathrm{Wh}(\pi_1(X))$. Note that this is independent of choice of base point of X, up to unique isomorphism.

The s-cobordism theorem (cf. [20], [21]) says that if M is a closed connected manifold of dimension at least 5 there is a one-to-one correspondence between $\mathcal{H}(M)$ and $\mathrm{Wh}(M)$ associating to the h-cobordism $(W; M, M')$ its Whitehead torsion $\tau(W; M) \in \mathrm{Wh}(M)$. Given an element $(W; M, M') \in \mathcal{H}(M)$ the restriction of a retraction $r : W \to M$ to M' is a homotopy equivalence $h : M' \to M$, uniquely determined up to homotopy. By a slight abuse of language, any such h will be referred to as "the natural homotopy equivalence". It induces a unique isomorphism

$$h_* : \mathrm{Wh}(M') \to \mathrm{Wh}(M).$$

Recall also that there is an involution $\tau \to \bar{\tau}$ on $\mathrm{Wh}(M)$ induced by transposition of matrices and inversion of group elements (cf. [21], [22]). If M is non-orientable, the involution is also twisted by the orientation character $\omega : \pi_1(M) \to \{\pm 1\}$, i.e. inversion of group elements is replaced by

$$\tau \mapsto \omega(\tau)\tau^{-1}. \tag{1}$$

Let $(W; M, M')$ and $(W; M', M)$ be dual h-cobordisms with M and M' of dimension n. Then $\tau(W; M)$ and $\tau(W; M')$ are related by the Milnor duality formula (cf. [21], [12])

$$h_*(\tau(W; M')) = (-1)^n \overline{\tau(W; M)}.$$

Definition 2.1. The *inertial set* of a closed connected manifold M is defined as

$$I(M) = \{(W; M, M') \in \mathcal{H}(M) | M \cong M'\},$$

or the corresponding subset of $\mathrm{Wh}(M)$.

There are many ways to construct inertial h-cobordisms. Here we recall three of these.

A. Let G be an arbitrary (finitely presented) group. Then there is a 2-dimensional finite simplicial complex K with $\pi_1(K) \cong G$. Let $\tau_0 \in \mathrm{Wh}(G)$ be an element with the property that $\tau_0 = \tau(f)$ for some homotopy self-equivalence $f : K \to K$. Denote by $N(K)$ a regular neighborhood of K in a high-dimensional Euclidean space $\mathbb{R}^n$ ($n \geqslant 5$ will do). By general position, we can approximate $f : K \to K \subseteq N(K)$