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Trees and generalised inverse limits on intervals

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We consider inverse limits of sequences of upper semicontinuous set-valued functions $f_{i+1} : \mathbb{I}_{i+1} \to 2^{\mathbb{I}_i}$ (where $\mathbb{I}_i = [0, 1]$ for each $i \in \mathbb{N}$), for which the graph of each bonding function is an arc. We show that any finite tree can be obtained as such an inverse limit, and one for which each bonding function is one of two specified functions. In addition, we discuss trees of height $\omega + 1$ that can be obtained as the inverse limit of such a sequence.

1. Introduction

Inverse limits have been vital in the development of continua theory as very complicated continua can be obtained as inverse limits with simple functions on simple spaces. Ingram and Mahavier recently introduced generalised inverse limits (GILs) [8,6], inverse limits of sequences of set-valued functions, and already GILs have demonstrated their considerable value as a new tool in the study of continua. For example, λ-dendroids can be obtained as GILs of simple bonding functions on closed intervals [7].

Research on GILs is being taken up by a growing number of authors. An extensive list of articles can be found in the books [5] by Ingram, and [6] by Ingram and Mahavier. One of the fundamental directions of enquiry is to understand which continua can be obtained as GILs on closed intervals, and why. Several classes of continua are not homeomorphic to a GIL whose system involves a single bonding function on intervals. For example, such an inverse limit can't be any of the following: a compact manifold of dimension greater than 1 [10]; a simple closed curve [4]; a finite graph other than an arc [11]; and a dendrite with

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a finite number of ramification points unless each ramification point has infinite degree and the set of them is contained in an arc [12]. It has only recently been established that there are continua that can't be obtained as a GIL if there is no restriction on the bonding functions, namely all compact 2-manifolds other that the torus [3].

Any planar continuum can be obtained as a GIL on closed intervals by letting the first graph in the sequence be one whose graph is the continuum, and all subsequent functions the identity (the function that maps each point x to $\{x\}$). A natural question is: can a planar continuum be obtained as a GIL if certain restrictions are put on the functions. We address the question: which trees can be obtained as a GIL on closed intervals for which the graph of each bonding function is an arc, in some sense the simplest extension of single-valued functions, and we show that any finite tree can be obtained as the GIL of such a sequence.

Young defined a set S of two bonding functions on intervals [13], and showed that every chainable continuum can be obtained as the inverse limit for which each bonding function is a member of S. Tree-like continua can be obtained as GILs on intervals and given our result it is reasonable to ask if any tree-like continuum can be obtained as a GIL for which the graph of each bonding function is an arc? If it is the case, then is there a finite set of functions on intervals whose graphs are arcs, such that every tree-like continuum can be obtained as a GIL with bonding functions members of this set? We define just two functions whose graphs are arcs, and given any finite tree T, we show how to construct a finite sequence of the two graphs whose Mahavier product is T, and how to construct a GIL for which each bonding function is one of the two. This provides a first step towards answering the latter question above.

We also discuss infinite trees and present examples of infinite trees that can be obtained as subsets of GILs for which the graphs of the bonding functions are arcs, and members of a finite set of graphs. Our examples include a full binary tree.

In Section 2 we give the definitions and define notation required in the sequel. In Section 3 we prove our main theorem: any finite tree can be obtained as the inverse limit of a sequence of upper semicontinuous functions for which the graph of each bonding function is an arc. In Section 4 we consider infinite trees.

2. Preliminaries

A *continuum* is a nonempty compact connected metric space. If X and Y are continua, the set of all nonempty closed subsets of Y is denoted 2^Y and a function $f : X \to 2^Y$ is called a *set-valued function*. A set-valued function $f : X \to 2^Y$ is *upper semicontinuous at a point* $x \in X$ if for every open set V in Y containing $f(x)$ there is an open set $U \subset X$ containing x such that for every $t \in U$, $f(t) \subseteq V$. The function f is called *upper semicontinuous* if it is upper semicontinuous at each point in X.

The *graph* of a function $f : X \to 2^Y$ is the set

$$\big\{(x, y) \in X \times Y \mid y \in f(x)\big\}.$$

It is well known that a function $f : X \to 2^Y$ is upper semicontinuous if and only if the graph of f is closed in $X \times Y$. We say that the graph of a function $f : X \to 2^Y$ is *surjective* if for each $y \in Y$ there is a point $x \in X$ such that $y \in f(x)$.

We focus on sequences of functions on $[0, 1]$. For each $i \in \mathbb{N}$ let $\mathbb{I}_i = [0, 1]$ and denote a sequence of upper semicontinuous functions $f_{i+1} : \mathbb{I}_{i+1} \to 2^{\mathbb{I}_i}$ by $\boldsymbol{f}$. We refer to a sequence $\boldsymbol{f}$ as an *inverse sequence*.

The *inverse limit* of an inverse sequence $\boldsymbol{f}$ is the set

$$\varprojlim \boldsymbol{f} = \left\{\boldsymbol{x} \in \prod_{i\in\mathbb{N}} \mathbb{I}_i \mid \forall\, i \in \mathbb{N}, x_i \in f_{i+1}(x_{i+1})\right\}$$

($\boldsymbol{x}$ denotes $(x_0, x_1, \ldots)$) with the subspace topology inherited from $\prod_{i\in\mathbb{N}} \mathbb{I}_i$.