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An accelerated symmetric SOR-like method for augmented systems☆

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ABSTRACT

Recently, Njeru and Guo presented an accelerated SOR-like (ASOR) method for solving the large and sparse augmented systems. In this paper, we establish an accelerated symmetric SOR-like (ASSOR) method, which is an extension of the ASOR method. Furthermore, the convergence properties of the ASSOR method for augmented systems are studied under suitable restrictions, and the functional equation between the iteration parameters and the eigenvalues of the relevant iteration matrix is established in detail. Finally, numerical examples show that the ASSOR is an efficient iteration method.

1. Introduction

Consider the following augmented systems of linear equations

$$\mathcal{A}z = \begin{pmatrix} A & B \\ -B^\top & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ -g \end{pmatrix}, \tag{1.1}$$

where $A \in \mathbb{R}^{m\times m}$ is a symmetric positive definite matrix, $B \in \mathbb{R}^{m\times n}$ is a matrix of full column rank, with $m \geq n$, $f \in \mathbb{R}^m$ and $g \in \mathbb{R}^n$ are given vectors, $B^\top$ denotes the transpose of the matrix B. The linear system (1.1) arises in many different applications of scientific and engineering applications, such as mixed finite element approximation to solve the Stokes and Navier–Stokes equations [1–3], computational fluid dynamics [4,5,38–46], constrained optimization [6], optimal control [7], and weighted least-squares problems [8,9], etc.

On account of the coefficient matrices A and B are large and sparse, iteration methods are becoming effective in storage requirements and preservation sparsity and even more attractive than direct methods for solving the augmented linear system (1.1). As we know, Young [10] established different variations of SOR method and studied its convergence properties in 1971. For solving the augmented system (1.1), Golub et al. [11] presented SOR-like method in 2001, Bai et al. [12] further proposed the generalized successive overrelaxation (GSOR) method and derived the optimal parameters in 2005, Darvishi and Hessari [13] studied the symmetric SOR (SSOR) method in 2006 and Wu et al. [14] further devised the MSSOR method in 2009. For more methods, see [15–19,27,28,37]. Moreover, Bai et al. [20,21] first proposed Hermitian and skew-Hermitian splitting (HSS) method and established the preconditioned HSS (PHSS) method. A multitude of the HSS methods can be

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found in [22–25]. In addition, Elman and Golub [26] discussed the inexact Uzawa method, Bai and Wang [29] proposed the parameterized inexact Uzawa (PIU) method for solving the augmented system (1.1). A great number of the Uzawa-type methods, see [30–32]. Other Krylov subspace methods such as preconditioned conjugate gradient (PCG) [33] and restrictively preconditioned conjugate gradient (RPCG) [34] iteration schemes and so on.

It is worth noting that Njeru and Guo [35] presented an accelerated SOR-like (ASOR) method and numerical results show that the convergence rates of the ASOR method faster than the SOR-like [11], GSOR [12] and GSSOR [36] methods by choosing optimal parameters. Note that the ASOR method takes the following iteration form:

$$\begin{cases} x^{k+1} = \dfrac{\alpha}{\alpha+\omega}x^k - \dfrac{\omega}{\alpha+\omega}A^{-1}(By^k - f) \\ y^{k+1} = y^k + \dfrac{2\omega}{2-\omega}Q^{-1}(B^\top x^{k+1} - g) \end{cases}. \tag{1.2}$$

In addition, the optimal experimental parameters of the ASOR method are given by

$$\alpha_{opt} = \frac{\sqrt{\mu_{\max}\mu_{\min}}}{8\sqrt{\mu_{\min}} + \sqrt{\mu_{\max}}}, \quad \omega_{opt} = \frac{2\sqrt{\mu_{\max}\mu_{\min}} + 1}{(\sqrt{\mu_{\max}} + \sqrt{\mu_{\min}})^2},$$

where $\mu_{\min}$ and $\mu_{\max}$ are the maximum and minimum eigenvalues of $Q^{-1}B^\top A^{-1}B$, respectively.

In this paper, we establish an accelerated symmetric SOR-like (ASSOR) method for solving the large and sparse augmented system (1.1), which has two real parameters α, ω. From the studied work of the ASOR method, we further discuss the convergence of the ASSOR method.

The rest of this paper is organized as follows. In Section 2, we develop an accelerated symmetric SOR-like (ASSOR) iteration method for solving augmented system (1.1). We discuss the convergence of the ASSOR method in Section 3, and some numerical computations are presented in Section 4. Finally, the paper is concluded in Section 5.

2. The accelerated symmetric SOR-like method

In this section, we propose the accelerated symmetric SOR-like (ASSOR) method for solving the augmented system (1.1).

Inspired by the idea of the ASOR [35] method, we make the following matrix splitting for the coefficient matrix $\mathcal{A}$ of the linear system (1.1) :

$$\mathcal{A} = \begin{pmatrix} A & B \\ -B^\top & 0 \end{pmatrix} = D - L - U \tag{2.1}$$

where

$$D = \begin{pmatrix} \alpha A & 0 \\ 0 & Q \end{pmatrix}, \ L = \begin{pmatrix} -A & 0 \\ B^\top & \frac{1}{2}Q \end{pmatrix}, \ U = \begin{pmatrix} \alpha A & -B \\ 0 & \frac{1}{2}Q \end{pmatrix},$$

and $\alpha \neq 0$ is a real number, $A \in \mathbb{R}^{m\times m}$ and $Q \in \mathbb{R}^{n\times n}$ are both symmetric positive definite matrices.

Let ω be a nonzero real number, $z^{(k)} = \left(x^{(k)^\top}, y^{(k)^\top}\right)^\top$ be the kth approximate solution of the system (1.1) and $u = (f^\top, -g^\top)^\top$. Thus, from the symmetric SOR [13] method, we obtain the following ASSOR iteration scheme:

$$(D - \omega L)z^{(k+\frac{1}{2})} = \big((1-\omega)D + \omega U\big)z^{(k)} + \omega u,$$

that is

$$z^{(k+\frac{1}{2})} = \mathcal{L}_{\alpha,\omega}z^{(k)} + \omega(D - \omega L)^{-1}u, \tag{2.2}$$

and

$$(D - \omega U)z^{(k+1)} = \big((1-\omega)D + \omega L\big)z^{(k+\frac{1}{2})} + \omega u,$$

which is equivalent to

$$z^{(k+1)} = \mathcal{U}_{\alpha,\omega}z^{(k+\frac{1}{2})} + \omega(D - \omega U)^{-1}u, \tag{2.3}$$

where

$$\begin{aligned} \mathcal{L}_{\alpha,\omega} &= (D - \omega L)^{-1}\big((1-\omega)D + \omega U\big) \\ &= \begin{pmatrix} \dfrac{\alpha}{\alpha+\omega}I_m & -\dfrac{\omega}{\alpha+\omega}A^{-1}B \\ \dfrac{2\alpha\omega}{(2-\omega)(\alpha+\omega)}Q^{-1}B^\top & I_n - \dfrac{2\omega^2}{(2-\omega)(\alpha+\omega)}Q^{-1}B^\top A^{-1}B \end{pmatrix}, \end{aligned}$$

and

$$\mathcal{U}_{\alpha,\omega} = (D - \omega U)^{-1}\big((1-\omega)D + \omega L\big)$$