Advances in Applied Mathematics 102 (2019) 83–112

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

Hankel determinants and shifted periodic continued fractions

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A R T I C L E   I N F O

Article history:
Received 26 June 2018
Received in revised form 3 September 2018
Accepted 20 September 2018
Available online xxxx

MSC:
primary 15A15
secondary 05A15, 11B83

Keywords:
Hankel determinants
Continued fractions

A B S T R A C T

Sulanke and Xin developed a continued fraction method that applies to evaluate Hankel determinants corresponding to quadratic generating functions. We use their method to give short proofs of Cigler's Hankel determinant conjectures, which were proved recently by Chang–Hu–Zhang using direct determinant computation. We find that shifted periodic continued fractions arise in our computation. We also discover and prove some new nice Hankel determinants relating to lattice paths with step set $\{(1,1),(q,0),(\ell-1,-1)\}$ for integer parameters m,q,ℓ. Again shifted periodic continued fractions appear.

1. Introduction

Let $A=(a_0,a_1,a_2\cdots)$ be a sequence, and denote by $A(x)=\sum_{n\geq 0} a_n x^n$ its generating function. Define the shifted Hankel matrices (or determinants) of A or $A(x)$ by

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https://doi.org/10.1016/j.aam.2018.09.004
0196-8858/

$$\mathcal{H}_n^{(k)}(A) = \mathcal{H}_n^{(k)}(A(x)) = (a_{i+j+k})_{0\le i,j\le n-1}, \qquad \text{and } H_n^{(k)}(A) = \det \mathcal{H}_n^{(k)}(A).$$

We shall write $H_n(A)$ for $H_n^{(0)}(A)$ and $H_n^1(A)$ for $H_n^{(1)}(A)$. In convention we set $H_0^{(k)} = 1$. If it is clear from the context, we will omit the A or $A(x)$. For instance, the sequence of Catalan number: 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862 $\cdots$ yields

$$\mathcal{H}_1 = [1], \mathcal{H}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \mathcal{H}_3 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 2 & 5 & 14 \end{bmatrix}, \mathcal{H}_4 = \begin{bmatrix} 1 & 1 & 2 & 5 \\ 1 & 2 & 5 & 14 \\ 2 & 5 & 14 & 42 \\ 5 & 14 & 42 & 132 \end{bmatrix}.$$

In recent years, a considerable amount of work has been devoted to Hankel determinants of path counting numbers, especially for weighted counting of lattice paths with up step $(1,1)$, level step $(\ell, 0)$, $\ell \ge 1$, and down step $(m-1,-1)$, $m \ge 2$. Many of such Hankel determinants have attractive compact closed formulas, such as that of Catalan numbers [17], Motzkin numbers [1,7], and Schröder numbers [3]. For instance, Motzkin numbers count lattice paths from $(0,0)$ to $(n,0)$ with step set $\{(1,1),(1,0),(1,-1)\}$ that never go below the horizontal axis. Partial Motzkin paths (similar to Motzkin paths but from $(a,0)$ to (n,b)) were considered in [9] and [16], where many nice determinant formulas were discovered.

Many methods have been developed for evaluating Hankel determinants using their corresponding generating functions. One of the basic tools is the method of continued fractions, either by J-fractions in Krattenthaler [15] or Wall [20] or by S-fractions in Jones and Thron [13, Theorem 7.2]. However, both of these methods need the condition that the determinant can never be zero, a condition not always present in our study.

Our point of departure is that such lattice paths have quadratic generating functions, so that Sulanke–Xin's continued fraction method applies to evaluate their Hankel determinants. In [18] Sulanke and Xin used Gessel–Xin's continued fraction method [12] to evaluate the Hankel determinants for lattice paths with step set $\{(1,1),(3,0),(1,-1)\}$.

Proposition 1. *[18] Let $F(x)$ be determined by $F(x) = 1 + x^3F(x) + x^2F(x)^2$. Then*

$$(H_n)_{n\ge 1}^{14} = (1,1,0,0,-1,-1,-1,-1,-1,0,0,1,1,1).$$

Moreover, if $m,n \ge 0$ with $(n-m) \equiv_{14} 0$, then $H_m = H_n$.

They indeed defined a quadratic transformation τ (see Proposition 6) such that there is a simple relation between the Hankel determinants of F and $\tau(F)$. Now if we let $F_0(x) = F(x)$ and apply the transformation τ, then we obtain the following periodic continued fractions:

$$F_0(x) \xrightarrow{\tau} F_1(x) \xrightarrow{\tau} \cdots \xrightarrow{\tau} F_5(x) = F_0(x). \tag{1}$$