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How to cut a cake with a Gram matrix

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ABSTRACT

In this article we study a problem of fair division. In particular we study a notion introduced by J. Barbanel that generalizes super envy-free fair division. We call this notion hyper envy-free. We give a new proof for the existence of such fair divisions. Our approach relies on classical linear algebra tools and allows us to give an explicit bound for this kind of fair division.
Furthermore, we also give a theoretical answer to an open problem posed by Barbanel in 1996. Roughly speaking, this question is: how can we decide if there exists a fair division satisfying some inequality constraints?
Furthermore, when all the measures are given with piecewise constant density functions then we show how to construct effectively such a fair division.

1. Introduction

In the following X will be a measurable set. This set represents an heterogeneous good, e.g. a cake, that we want to divide between n players. A division of the cake is a partition $X = \sqcup_{i=1}^{n} X_i$, where each X_i is a measurable subset of X. After the

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division X_i is given to the i-th player. A natural and old problem is: how to get a fair division?

This problem appears when we study division of land, time or another divisible resource between different agents with different points of view. These problems appear in the mathematics, economics, political science, artificial intelligence and computer science literature, see [8,11,6].

In order to study this problem, to each player is associated a non-atomic probability measure μ_i. Thus, in particular $\mu_i(X) = 1$, and $\mu_i(A \sqcup B) = \mu_i(A) + \mu_i(B)$, where A, and B are disjoint measurable sets. These measures represent the preference of each player. Several notions of fair divisions exist:

- Proportional division: $\forall i$, $\mu_i(X_i) \geq 1/n$.
- Exact division: $\forall i$, $\forall j$, $\mu_i(X_j) = 1/n$.
- Equitable division: $\forall i$, $\forall j$, $\mu_i(X_i) = \mu_j(X_j)$.
- Envy-free division: $\forall i$, $\forall j$, $\mu_i(X_i) \geq \mu_i(X_j)$.

All these fair divisions are possible, see e.g. [14,8,5,12,7,13,16].

Some fair divisions are possible under some conditions. This is the case for super envy-free fair divisions. We recall the definition of a super envy-free fair division:

- Super envy-free division: $\forall i$, $\forall j \neq i$, $\mu_i(X_i) > 1/n > \mu_i(X_j)$.

Barbanel has shown in [2] that a super envy-free division is possible if and only if the measures μ_i are linearly independent. Actually, if the measures are linearly independent then there exists a real $\delta > 0$ such that:

$$\mu_i(X_i) \geq 1/n + \delta, \text{ and } \mu_i(X_j) \leq 1/n - \delta/(n-1).$$

We can define an even more demanding fair division. For example, we can imagine that the first player would like to get a partition such that:

$$\begin{aligned}
&\mu_1(X_1) = 1/n + 3\delta,\\
&\mu_1(X_3) = 1/n + 2\delta,\\
&\mu_1(X_4) = \mu_1(X_5) = 1/n + \delta,\\
&\mu_1(X_2) = \mu_1(X_6) = 1/n - 6\delta.
\end{aligned}$$

This means that the third, the fourth and the fifth player are friends with the first player, but the second and sixth player are not friends with this player. Furthermore, the first player prefers the third to the fourth and the fifth player. Thus the previous conditions given by the first player imply:

$$\mu_1(X_1) > \mu_1(X_3) > \mu_1(X_4) = \mu_1(X_5) > \frac{1}{n} > \mu_1(X_2) = \mu_1(X_6).$$