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SET-DIRECT FACTORIZATIONS OF GROUPS

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ABSTRACT. We consider factorizations $G = XY$ where G is a general group, X and Y are normal subsets of G and any $g \in G$ has a unique representation $g = xy$ with $x \in X$ and $y \in Y$. This definition coincides with the customary and extensively studied definition of a direct product decomposition by subsets of a finite abelian group. Our main result states that a group G has such a factorization if and only if G is a central product of $\langle X \rangle$ and $\langle Y \rangle$ and the central subgroup $\langle X \rangle \cap \langle Y \rangle$ satisfies certain abelian factorization conditions. We analyze some special cases and give examples. In particular, simple groups have no non-trivial set-direct factorization.

1. INTRODUCTION

Factorizations of groups is an important topic in group theory that has many facets. The most basic and best studied type of factorization is the direct product factorization of a group into two normal subgroups. If G is a group and H and K are two normal subgroups of G then $G = H \times K$ if and only if each $g \in G$ has a unique representation $g = hk$ with $h \in H$ and $k \in K$. One possibility to generalize this definition is to relax the condition that both H and K are normal. This leads to the well-known concept of a semi-direct product of subgroups (only one of the factors is assumed to be normal) and also to the consideration of factorizations $G = HK$ where neither of the two subgroups H and K is normal, and even the unique representation condition is not necessarily assumed. To get a glimpse of the possibilities see the seminal classification result in [11] of maximal decompositions $G = HK$ where G is a finite simple group and H and K are two maximal subgroups of G.

Another generalization arose from a geometry problem of Minkowski [13]. In 1938 Hajós [7] reformulated this problem as an equivalent factorization problem of a finite abelian group, where the factors need not be subgroups. More precisely, if G is an abelian group written additively then a (set) factorization of G is a representation of G in the form $G = H + K$ where H and K are subsets of G and for each $g \in G$ there is a unique pair $h \in H$ and $k \in K$ such that $g = h + k$. Four years later, Hajós [8] solved Minkowski's problem by solving the equivalent group factorization problem. This initiated the investigation of set factorizations

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