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The Hilton–Milner theorem for finite affine spaces

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ABSTRACT

Suppose $3 \leq 2k+1 < n$. Let $AG(n,\mathbb{F}_q)$ be the n-dimensional affine space over the finite field $\mathbb{F}_q$ and $\mathcal{M}(k,n)$ be the set of all k-flats in $AG(n,\mathbb{F}_q)$. Suppose that $\mathcal{F} \subseteq \mathcal{M}(k,n)$ is an intersecting family with $\bigcap_{F\in\mathcal{F}} F = \emptyset$. In this paper, we show $|\mathcal{F}| \leq 1 + {n-1 \brack k-1}_q + \sum_{i=0}^{k-1} q^{i(i+1)}(q^{k-i}-1){n-1-k \brack i}_q {k \brack i}_q$, and describe the structure of $\mathcal{F}$ which reaches this bound.

1. Introduction

For positive integers k and n with $k \leq n$, let $[n] = \{1, 2, \dots, n\}$ and $\binom{[n]}{k}$ be the collection of all k-subsets of $[n]$. A family $\mathcal{F} \subseteq \binom{[n]}{k}$ is called *intersecting* if $|A \cap B| \geq 1$ for all $A, B \in \mathcal{F}$. Erdős, Ko and Rado [5] determined the maximum size of an intersecting family.

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Theorem 1.1 *(Erdős–Ko–Rado). Let $n \geq 2k+1$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ be an intersecting family. Then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. Equality holds only if $\mathcal{F} = \{A \in \binom{[n]}{k} \mid i \in A\}$ for some $i \in [n]$.*

For any family $\mathcal{F} \subseteq \binom{[n]}{k}$, the *covering number* $\tau(\mathcal{F})$ is the minimum size of a subset of $[n]$ that meets all $A \in \mathcal{F}$. From Theorem 1.1 we deduce that if $n \geq 2k+1$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is an intersecting family of maximum size, then $\tau(\mathcal{F}) = 1$.

Let $\mathcal{F} \subseteq \binom{[n]}{k}$ be an intersecting family with $\tau(\mathcal{F}) \geq 2$. Hilton and Milner [11] determined the maximum size of $\mathcal{F}$. Frankl and Füredi [6] gave a different proof for the same result.

Theorem 1.2 *(Hilton–Milner). Let $n \geq 2k+1 \geq 5$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ be an intersecting family with $\tau(\mathcal{F}) \geq 2$. Then $|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$. Equality holds only if*

(i) *$\mathcal{F} = \{A\} \cup \{B \in \binom{[n]}{k} \mid i \in B, A \cap B \neq \emptyset\}$ for some k-subset A of $[n]$ and $i \in [n] \setminus A$.*
(ii) *$\mathcal{F} = \{B \in \binom{[n]}{3} \mid |A \cap B| \geq 2\}$ for some 3-subset A of $[n]$ if $k = 3$.*

This theorem is called the Hilton–Milner theorem now. Over the years, there have been many intersecting extensions for this result. See [1] for vector spaces, [8] for bilinear forms graphs, [13] for set partitions, and [14] for weak compositions.

Let $\mathbb{F}_q$ be the finite field with q elements, where q is a prime power. Let $\mathbb{F}_q^n$ be the n-dimensional row vector space over $\mathbb{F}_q$ and $\begin{bmatrix}[n]\\ m\end{bmatrix}$ be the set of all m-dimensional subspaces of $\mathbb{F}_q^n$, where $0 \leq m \leq n$. For brevity we use the *Gaussian coefficient*

$$\begin{bmatrix}n\\ k\end{bmatrix}_q = \prod_{i=n-k+1}^{n}(q^i - 1)/\prod_{i=1}^{k}(q^i - 1).$$

By convenience $\begin{bmatrix}n\\ 0\end{bmatrix}_q = 1$ and $\begin{bmatrix}n\\ k\end{bmatrix}_q = 0$ whenever $k < 0$ or $n < k$. Then the size of $\begin{bmatrix}[n]\\ m\end{bmatrix}$ is $\begin{bmatrix}n\\ m\end{bmatrix}_q$. A family $\mathcal{F} \subseteq \begin{bmatrix}[n]\\ k\end{bmatrix}$ is called *intersecting* if $\dim(A \cap B) \geq 1$ for all $A, B \in \mathcal{F}$. Hsieh [12] determined the maximum size of an intersecting family. For different proofs of the same result, see [4] and [7].

Theorem 1.3 *(Hsieh). Let $n \geq 2k+1$ and $\mathcal{F} \subseteq \begin{bmatrix}[n]\\ k\end{bmatrix}$ be an intersecting family. Then $|\mathcal{F}| \leq \begin{bmatrix}n-1\\ k-1\end{bmatrix}_q$. Equality holds only if $\mathcal{F} = \{A \in \begin{bmatrix}[n]\\ k\end{bmatrix} \mid E \subseteq A\}$ for some subspace $E \in \begin{bmatrix}[n]\\ 1\end{bmatrix}$.*

For any family $\mathcal{F} \subseteq \begin{bmatrix}[n]\\ k\end{bmatrix}$, the *covering number* $\tau(\mathcal{F})$ is the minimum dimension of a subspace of $\mathbb{F}_q^n$ that intersects all $A \in \mathcal{F}$. From Theorem 1.3 we deduce that if $n \geq 2k+1$ and $\mathcal{F} \subseteq \begin{bmatrix}[n]\\ k\end{bmatrix}$ is an intersecting family of maximum size, then $\tau(\mathcal{F}) = 1$.

Let $\mathcal{F} \subseteq \begin{bmatrix}[n]\\ k\end{bmatrix}$ be an intersecting family with $\tau(\mathcal{F}) \geq 2$. Blokhuis et al. [1] determined the maximum size of $\mathcal{F}$.