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A new mathematical formulation for a phase change problem with a memory flux

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ABSTRACT

A mathematical formulation for a one-phase change problem in a form of Stefan problem with a memory flux is obtained. The hypothesis that the integral of weighted backward fluxes is proportional to the gradient of the temperature is considered. The model that arises involves fractional derivatives with respect to time both in the sense of Caputo and of Riemann–Liouville. An integral relation for the free boundary, which is equivalent to the "fractional Stefan condition", is also obtained.

1. Introduction

The theory related to heat diffusion has been extensively developed in the last century. Modelling classical heat diffusion comes hand in hand with Fourier Law. Nevertheless, we shall not forget that this famous law is an experimental phenomenological principle.

In the past 40 years, many generalized flux models of the classical one (i.e. the one derived from Fourier Law) were proposed in the literature and accepted by the scientific community. See e.g. [8,16–18,37].

In this paper a phase change problem for heat diffusion under the hypothesis that the heat flux is a flux with memory is analysed. This kind of problems are known in the literature as Stefan problems [41,42].

The model obtained under the memory assumption is known as an anomalous diffusion model, and it is governed by fractional diffusion equations. There is a vast literature in the subject of fractional diffusion equations. We refer the reader to [24,30,31] and references therein.

The study of anomalous diffusion has its origins in the investigation of non-Brownian motions (Random walks). In that context it was observed that "the mean square displacement" of the particles is proportional to a power of the time, instead of being proportional just to time. An exhaustive work in this direction has been done by Metzler and Klafter [26]. Other articles in this direction are [19,25,27,28]. It is worth mentioning that many works (see e.g. [3,12,40]) suggest that the anomalous diffusion is caused by heterogeneities in the domain.

Before presenting the problem, let us establish some usual notation related to heat conduction with the corresponding physical

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dimensions. Let us write **T** for temperature, **t** for time, **m** for mass and **X** for position.

$$\begin{array}{lll} u & \textit{temperature} & [\mathbf{T}] \\ k & \textit{thermal conductivity} & \left[\frac{\mathbf{m\ X}}{\mathbf{Tt}^3}\right] \\ \rho & \textit{mass density} & \left[\frac{\mathbf{m}}{\mathbf{X}^3}\right] \\ c & \textit{specific heat} & \left[\frac{\mathbf{X}^2}{\mathbf{Tt}^2}\right] \\ d=\frac{k}{\rho c} & \textit{diffusion coefficient} & \left[\frac{\mathbf{X}^2}{\mathbf{t}}\right] \\ l & \textit{latent heat per unit mass} & \left[\frac{\mathbf{X}^2}{\mathbf{t}^2}\right] \end{array} \tag{1}$$

Consider a temperature function $u = u(x, t)$ and its corresponding flux $J(x, t)$, both defined for a semi-infinite unidimensional material. From the First Principle of Thermodynamics, we deduce the continuity equation

$$\rho c\frac{\partial u}{\partial t}(x,t) = -\frac{\partial J}{\partial x}(x,t). \tag{2}$$

The aim of this work is to derive a model by considering a special non-local memory flux. For example, Gurtin and Pipkin [15] (experts in continuum mechanics and heat transfer) proposed in 1968 a general theory of heat conduction with finite velocity waves through the following non local flux law:

$$J(x,t) = K(t) * \left(-k\frac{\partial}{\partial x}u(x,t)\right) = -k\int_{-\infty}^{t} K(t-\tau)\frac{\partial}{\partial x}u(x,\tau)\mathrm{d}\tau, \tag{3}$$

where K is a positive decreasing kernel which verifies $K(s) \to 0$ when $s \to \infty$.

Let us comment on some different explicit and implicit definitions of fluxes, and their effects on the resulting governing equations:

- **Explicit forms for the flux:** $J(x,t) = F(x,t)$
  The classical law for the flux is the *Fourier Law*, which states that the flux J is proportional to the temperature gradient, that is:
  $$J(x,t) = -k\frac{\partial}{\partial x}u(x,t). \tag{4}$$
  If alternatively suppose that the flux at the point (x, t) is proportional to the total flux, then the given law is the following:
  $$J(x,t) = \frac{1}{\tilde{\tau}}\int_{-\infty}^{t} -k\frac{\partial}{\partial x}u(x,\tau)\mathrm{d}\tau. \tag{5}$$
  In (5), $\tilde{\tau}$ is a constant whose physical dimension is time. Another interesting thing is that (5) can be interpreted as a generalized sum of backward fluxes, where every local flux has the same "relevance".
  The following expression for the flux is a generalized sum of weighted backward fluxes. There is now a kernel which assigns more weight ("importance") to the nearest temperature gradients, that is:
  $$J(x,t) = -\frac{\eta_\alpha}{\Gamma(\alpha)}\int_{-\infty}^{t}(t-\tau)^{\alpha-1}k\frac{\partial}{\partial x}u(x,\tau)\mathrm{d}\tau. \tag{6}$$
  Here, α is a constant in the interval (0,1) that plays an important role, and η_α is a constant imposed to equate units of measures. Both will be specified later.
  Note that (4) and (6) result from considering the kernels $K_1(t) \equiv \delta(t)$ and $K_2(t) = \eta_\alpha \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, respectively, in the generalized flux equation (3).
- **Implicit forms for the flux:** $F(x,t,J(x,t)) = G(x,t)$.
  One of the most famous formulations for the flux, is given by the Cattaneo's equation [6]
  $$J(x,t) + \tilde{\tau}\frac{\partial}{\partial t}J(x,t) = -k\frac{\partial}{\partial x}u(x,t), \tag{7}$$
  which was proposed with the aim of introducing an alternative to the "unphysical" property of the diffusion equation known as *infinite speed of propagation*. Eq. (7) can be seen as a first order Taylor approximation of (8) in which the flux is allowed to adjust to the gradient of the temperature according to a relaxation time $\tilde{\tau}$,
  $$J(x,t+\tilde{\tau}) = -k\frac{\partial}{\partial x}u(x,t). \tag{8}$$
  Another approach assumes that the integral of the back fluxes, at the current time, is proportional to the gradient of the temperature:
  $$\frac{1}{\tilde{\tau}}\int_{-\infty}^{t} J(x,\tau)\mathrm{d}\tau = -k\frac{\partial}{\partial x}u(x,\tau).$$
  Yet another formulation considers that *the integral of the weighted backward fluxes at the current time, is proportional to the gradient of the temperature*:
  $$\frac{\nu_\alpha}{\Gamma(1-\alpha)}\int_{-\infty}^{t}(t-\tau)^{-\alpha}J(x,\tau)\mathrm{d}\tau = -k\frac{\partial}{\partial x}u(x,\tau). \tag{9}$$

Note 1. Although when we talk about backward fluxes it is logical to consider the lower limit of the integral at $-\infty$, we can suppose that the function u has remained constant (for some reason) for all $t<0$, where with 0 we refer to a certain initial time. Moreover, under this condition, that is $u(x, t) \equiv u_0$, for every $t<0$, the expressions (6) and (9) become

$$J(x,t) = -\frac{\eta_\alpha}{\Gamma(\alpha)}\int_{0}^{t}(t-\tau)^{\alpha-1}k\frac{\partial}{\partial x}u(x,\tau)\mathrm{d}\tau, \tag{10}$$

and

$$\frac{\nu_\alpha}{\Gamma(1-\alpha)}\int_{0}^{t}(t-\tau)^{-\alpha}J(x,\tau)\mathrm{d}\tau = -k\frac{\partial}{\partial x}u(x,\tau), \tag{11}$$

respectively.

Expressions (10) and (11) are closely linked to fractional calculus. Let us present the basic definitions that will be employed throughout the article.

Definition 1. Let $[a, b] \subset \mathbb{R}$ and $\alpha \in \mathbb{R}^+$ be such that $n-1 < \alpha \le n$.

1. For $f \in L^1[a, b]$, we define the *fractional Riemann–Liouville integral of order* α as
   $${}_aI^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\tau)^{\alpha-1}f(\tau)\mathrm{d}\tau.$$
2. For $f \in AC^n[a,b] = \left\{f \mid f^{(n-1)} \text{ is absolutely continuous on [a,b]}\right\}$, we define the *fractional Riemann–Liouville derivative of order* α as
   $${}^{RL}_{a}D^\alpha f(t) = \left[D^n\, {}_aI^{n-\alpha}f\right](t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t (t-\tau)^{n-\alpha-1}f(\tau)\mathrm{d}\tau.$$
3. For $f \in W^n(a,b) = \left\{f \mid f^{(n)} \in L^1[a,b]\right\}$, we define the *fractional Caputo derivative of order* α as
   $${}^{C}_{a}D^\alpha f(t) = \left[{}_aI^{n-\alpha}(D^n f)\right](t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)}\displaystyle\int_a^t (t-\tau)^{n-\alpha-1}f^{(n)}(\tau)\mathrm{d}\tau, & n-1<\alpha<n, \\ f^{(n)}(t), & \alpha = n. \end{cases}$$