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# Self-oscillation excitation under condition of positive dissipation in a state-dependent potential well

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ABSTRACT

The self-oscillatory dynamics is considered as motion of a particle in a potential field in the presence of dissipation. Described mechanism of self-oscillation excitation is not associated with peculiarities of a dissipation function, but results from properties of a potential, whose shape depends on a system state. Moreover, features of a potential function allow to realize the self-oscillation excitation in a case of the dissipation function being positive at each point of the phase space. The phenomenon is explored both numerically and experimentally on the example of a double-well oscillator with a state-dependent potential and dissipation. After that a simplified single-well model is studied.



## 1. Introduction

The problem of the self-sustained oscillation existence in autonomous dynamical systems was originally introduced by Poincare [1]. Thereafter bifurcation mechanisms of self-sustained oscillation excitation were described by Andronov and his associates [2] and Hopf [3,4]. Despite the issue of the self-sustained oscillations is known for over a century, this topic is still attractive and interesting. This is due to the fact that self-sustained oscillators are of a frequent occurrence in physics [5–8], chemistry [9–11], geology [12,13], climatology [14–16], biology [17–20], economics [21,22] and other fields. Problematics of self-oscillations in deterministic systems includes regular periodical motions as well as the chaotic dynamics [23–25]. There are well-known stochastic effects, which are related to the self-oscillation topic: stochastic bifurcations (in the context of the stochastic Andronov–Hopf bifurcation [26–30]), coherence resonance [31–36], stochastic synchronization [37–39], stochastic resonance [40,41]. Since a class of the self-oscillatory systems includes a broad variety of dynamical systems with different nature and features, the problem of general description and interpretation is significantly important. This issue will be discussed in the present paper.

The dynamics of an oscillator with one degree of freedom can be interpreted as motion of a particle in a potential field in the presence of dissipation. In that case a mathematical model of the oscillator is presented in the following form:

$$\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \frac{dU}{dy} = 0, \tag{1}$$

where the factor $\gamma$ characterizes dissipation, $U$ is a function denoting the potential field. Typically, the potential function is assumed to be univariate: $U = U(y)$. Depending upon the specific of the dynamical system (1), the dissipation factor $\gamma$ can be either a fixed parameter, $\gamma = const$, or a state-dependent function $\gamma = \gamma(y, \frac{dy}{dt})$. In the presence of constant dissipation ($\gamma = const$) the deterministic model (1) exhibits two kinds of the behaviour in the potential well: either damped oscillation in case $\gamma > 0$ or oscillations with unlimited amplitude growth in case $\gamma < 0$. If the dissipation depends on a system state and the function $\gamma(y, \frac{dy}{dt})$ possesses negative values in some area of the phase space, self-oscillation excitation can be achieved. In such a case the self-oscillation excitation is a result of the negative dissipation action, which is associated with pumping of energy.

The described principle is simply illustrated on the example of the Van der Pol self-sustained oscillator [42], which is described by the following equation:

$$\frac{d^2y}{dt^2} + \left(y^2 - \varepsilon\right)\frac{dy}{dt} + y = 0, \tag{2}$$

where $y$ is the dynamical variable, $\varepsilon$ is the parameter, which determines the dynamics. Autonomous system (2) exhibits the regular quiescent or self-oscillatory dynamics, while the non-autonomous model can realize singular solutions [43]. In terms of motion of a particle in a potential field the system (2) describes oscillations in the potential $U(y) = \frac{1}{2}y^2 + K$ (the constant $K$ is neglected in the

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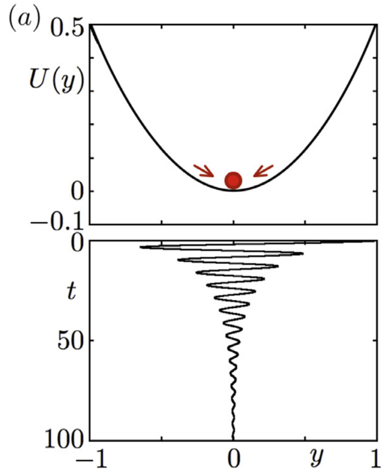


**Fig. 1.** System (2). Description of the dynamics as motion of a particle in the potential field $U(y)=\frac{1}{2}y^2$ (upper panels) and corresponding time realization $y(t)$ (lower panels) at $\varepsilon=-0.1$ (a) and $\varepsilon=0.1$ (b). The grey area corresponds to negative values of the dissipation function $\gamma(y)$.

following, $K=0$) in the presence of dissipation defined by the factor $\gamma=\gamma(y)=y^2-\varepsilon$. In case $\varepsilon<0$ the dissipation function $\gamma(y)$ is positive at any values of $y$. Then oscillations $y(t)$ are damped and all trajectories are attracted to a stable equilibrium point in the origin, corresponding to a potential well bottom (Fig. 1(a)). If $\varepsilon>0$, the function $\gamma(y)$ possesses negative values in the vicinity of a potential well bottom (the grey area in Fig. 1(b)) and the equilibrium point in the origin becomes unstable. The phase point is repelled from the unstable steady state and goes out of the area corresponding to negative $\gamma(y)$. After that the amplitude growth slows down and stops. After transient time stationary periodical self-oscillations are organized. There is energy balance between dissipation and pumping during the period of the self-oscillations. The same principle of self-oscillation excitation takes place in the Rayleigh [44] self-oscillator, FitzHugh–Nagumo model [45,46] and in other examples of self-sustained oscillators.

Another principle of self-oscillation excitation is described in the current paper. As will be shown below, the self-oscillation excitation can be realized in the system (1) in a case of the dissipation factor $\gamma$ being positive at each point of the phase space. This phenomenon becomes possible due to specific features of the potential function assumed to be bivariate, $U=U(y,\frac{dy}{dt})$. In other words, the problem of motions of the point mass in a state-dependent potential field with positive dissipation will be considered.

At first, the explored issue will be numerically and experimentally studied in the context of a model of a double-well oscillator with state-dependent dissipation and potential, which exhibits hard oscillation excitation. Then a simplified model, Eq. (1) with the positive drag coefficient $\gamma>0$, which demonstrates soft self-oscillation excitation, will be proposed. It has to be noted that the used term "self-oscillation excitation under condition of positive dissipation" is correct only in the context of motion of a particle in a potential field. Generally, dissipativity of the considered systems is characterized by the divergence of the phase space flow. The divergence corresponding to the self-oscillatory dynamics in the studied models is positive in some regions of the phase space, while it is negative in other areas. In this point of view the occurrence of self-oscillatory behaviour traced by a stable limit cycle is understandable and logical.

Numerical modelling of the studied systems was carried out by integration using the Heun method [47] with the time step $\Delta t=0.0001$.

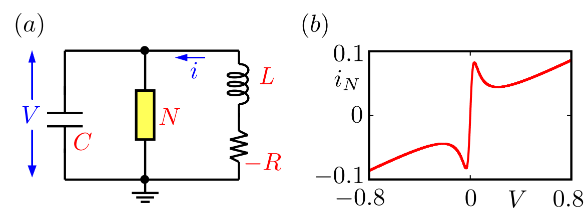


**Fig. 2.** (a) Schematic circuit diagram of double-well oscillator with nonlinear dissipation; (b) Current–voltage characteristic of element N corresponding to parameters $a=200$, $b=0.2$, $g=0.1$.

## 2. Self-oscillatory motion in a double-well state-dependent potential field in the presence of positive dissipation

Fig. 2(a) shows the system under study. It is a simplified modification of the bistable self-oscillator, offered in the paper [41]. Fig. 2(a) shows a parallel oscillatory circuit including the negative resistance -R and the nonlinear element N with the current–voltage characteristic $i_N(V)=\frac{V}{aV^2+b}+gV$ (Fig. 2(b)). By using the Kirchhoff's current law the following differential equations for the voltage $V$ across the capacitor $C$ and the current $i$ through the inductor $L$ can be derived:

$$\begin{cases} C\dfrac{dV}{dt'}+i+\dfrac{V}{aV^2+b}+gV=0, \\ V=L\dfrac{di}{dt'}-Ri. \end{cases} \quad (3)$$

In the dimensionless variables $x=V/V_0$ and $y=i/i_0$ with $V_0=1$ V, $i_0=1$ A and dimensionless time $t=[(V_0/(i_0L)]t'$, Eq. (3) can be re-written as,

$$\begin{cases} \varepsilon\dot{x}=-y-gx-\dfrac{x}{ax^2+b}, \\ \dot{y}=x+my, \end{cases} \quad (4)$$

where $\dot{x}=\frac{dx}{dt}$, $\dot{y}=\frac{dy}{dt}$, the parameter $\varepsilon$ sets separation of slow and fast motions, other parameters are $g$, $a$, $b$, $m>0$. Eq. (4) can be written in the "coordinate-velocity" form with the dynamical variables $y$, $v\equiv\dot{y}$:

$$\begin{cases} \dot{y}=v, \\ \varepsilon\dot{v}=y(mg-1)+v(\varepsilon m-g) \\ +\dfrac{my-v}{a(v-my)^2+b}. \end{cases} \quad (5)$$